On the Emergence of Cities

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Abstract

This paper contains a description of a general class of city formation models. Individual economic agents have preferences for locations that depend upon the population distribution. A location's attractiveness depends upon some combination of its population and its average distance to other agents. Economic variables enter indirectly. Taking this broad perspective leads to a deeper understanding of how cities form as well as of the sensitivity to initial conditions of their locations and sizes. In addition, this class of models supports scenarios where cities emerge: without any assumptions that agents wish to live near or with one another, agents cluster into cities.

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1 Introduction

In this paper, I analyze city formation by constructing a class of models where agents' preferences depend on a location's population and its separation (its average distance to other agents). Taking this minimalist modeling approach increases our understanding of how individual preferences over population distributions map into final configurations. While assuming preferences over population distributions omits economic variables explicitly, so long as agents' preferences are consistent with the economic forces that cause cities to form, the gains from abstraction can outweigh the costs. And, if you strip the process of city formation down to its bare essentials, population and separation seem to be the most relevant characteristics. Population captures multiple economic forces. Prices, space (O'Hara [22]), range of goods (Krugman [19]), externalities (Arthur [1]), city characteristics, public goods, and per capita market setup costs (Berliant and Konishi [3]) all depend upon city size. Separation also serves as a proxy for multiple economic forces, several of which stem either directly or indirectly from transportation costs. Clearly, commuting time and costs depend on distance. Commodity prices and wages within a city also vary with the extent of competition from other cities which in turn depends on distance. Separation may also capture a desire to avoid city generated negative externalities such as pollution, crime, and traffic, which dissipate with distance.

Not surprisingly, city size and transportation costs have been prominent in the city formation literature since its inception (see Christaller [4] and Henderson [10]). And, as would be expected, they continue to play large roles in present models. Fujita [7] includes both transportation costs and positive externalities, and Krugman [17], whose model features imperfect competition among producers of differentiated products, also relies on transportation costs and pecuniary externalities to generate cities.

I allow for preferences over population and separation to be positive or negative, and linear or nonlinear, enabling an investigation into what causes cities to form. If everyone wants to maximize the population at their home location or to minimize their separation from others, then they form a single city. Much less transparently, if agents prefer to maximize their distance from others, they too form cities. In this case, cities emerge (Forrest [6]). This result contradicts the intuition that for cities to exist there must be some external benefits from agglomeration (Lucas [20]).[1] I also analyze the sensitivity of city location and the city size distribution to initial conditions. The difference is crucial. Sensitivity of the former type occurs far more frequently than does the latter. This implies that it might be easier to find regularities in the size of cities than in their locations.

[1] Many others besides Lucas have made this claim. For example, Thisse [26] states "In order to account for the polarization of space, one must first recognize that increasing returns are essential for explaining the geographical distribution of production activities." (pg 299). These preferences for agglomeration need not be large. Papageorgiou and Smith [24] derive rather mild conditions on the externalities between locations necessary for agglomeration.

Though this paper discusses city formation within a country, several of the results can be interpreted as describing the location of firms and people within a city.[2] For example, O'Hara [22] constructs a model of a central business district on a square that relies on the same separation measure that I use here. His model differs in that he includes land rents, an economic variable that I omit. Also, Beckman [2] analyzes locational choice within a city. In his model, agent utility depends upon average distance, space, and consumption. Only the last of these is missing from my model, and it enters his model linearly. That said, to call these models special cases of my model would be incorrect. They provide richer micro foundations. Instead, my model should be interpreted as providing an organizing perspective through which these separate contributions can be interpreted.

A general model that captures the salient features of the city formation process has value given the renewed emphasis on cities within economics. This resurgence stems from a combination of intellectual forces. On the one hand, the engaging historical accounts of Jacobs ([13] and [14]) and Cronon [5] and a recognition of the importance of increasing returns in the economy writ large (Arthur [1]) have led economists to re-evaluate the role of cities in creating economic growth. As more economists come to accept that cities play a central role in the economic fortunes of a country, i.e. that as in Jacobs' theory they form the nucleus of the atom, then how and where cities form and how large cities should be and will tend to be become important questions. On the other hand, simultaneous advancements in endogenous growth theory and formal models of imperfect competition have enabled economists to construct formal models of city formation, a point elaborated by Krugman [18].

Many of these recent models include individuals and firms with preferences over economic variables: wages, land prices, the array of goods, etc., others define preferences over the population distribution as I do here. A third alternative creates variables, such as market potential, that are neither features of the population distribution nor economic variables (Harris [9]. Not surprisingly, market potential models have been criticized for lacking economic underpinnings. While valid, this criticism does not necessarily extend to models that depend on spatial variables. The distinction between economic and spatial variables can be blurry. Assumptions on economic variables that yield positive externalities from agglomeration generate preferences where people prefer to live in locations with greater populations. In some cases, it may be more elegant (and tractable) to just assume a preference for population. People who desire the externalities generated by agglomeration might rank cities by size. Whether size then becomes an economic variable becomes a semantic question.

The remainder of this paper proceeds as follows: Section 2 contains the basic model of location choice and section 3 presents mathematical and computational results for the eight cases with linear preferences, In section 4, I analyze two nonlinear cases. The discussion at the end of the paper addresses possible extensions of the

[2] I would like to thank a referee for making this suggestion.

model.

2 A Model of City Formation

The model is purposefully simple so that it captures how spatial population accumulation depends upon agents' preferences. A finite number of agents (individuals or firms [3]) reside on an N by N lattice. I do *not* connect the edges of the lattice to form a torus.[4] The assumption of a square lattice matters. Several findings depend upon it. This shortcoming appears unavoidable and suggests the need for future work with irregular lattices and lattices approximating actual geography.

Def'n: *The set of* **agents** $M = \{1, 2, 3, ...m\}$

The set of all possible locations is an N by N lattice.

Def'n: *The set of* **locations** $N \times N$, *where* $N = \{1, 2, 3, ...n\}$

Agents choose locations. More than one agent can occupy the same location on the lattice. An agent's utility level, and, perforce, her location decision depend upon the distribution of agents on the lattice. Let F denote a distribution of agents on the set of locations. F implicitly maps the set of locations into M so that F_{ij} denotes the number of agents residing in the ith row and jth column of the lattice.

Def'n: *The set of* **distributions** *of agents*

$$\Psi = \{F : N \times N \to M \cup \{0\}, \text{ and } \sum_{i=1}^{n} \sum_{j=1}^{n} F_{ij} = m\}$$

Agents have identical preferences over the distributions of agents. An agent's utility depends upon her own location, and on the entire distribution.

Def'n: *The* **utility function** $u : N \times N \times F \to \Re$

Given identical agents, a distribution F can be considered *utility maximizing* if it maximizes a utilitarian social welfare function. All agents have identical preference, implying that a utility maximizing distribution is also Pareto Efficient.

Def'n: *The distribution* $F \in \Psi$ **utility maximizing** *if and only if*

$$\sum_{i=1}^{n} \sum_{j=1}^{n} F_{ij} \cdot u(i, j, F) \geq \sum_{i=1}^{n} \sum_{j=1}^{n} F'_{ij} \cdot u(i, j, F') \quad \textit{for all } F' \in \Psi$$

[3]In several of the scenarios considered, firms may be a more appropriate interpretation.

[4]I also do not consider the case where agents locate on the surface of a sphere. Although the earth is round, most countries are topologically equivalent to the square.

An agent's utility depends upon at most two characteristics of the population distribution: the population at her home location and the average distance from her home location to the other agents. Many underlying economic variables generate preferences that are consistent with home location's population entering utility positively. Technological externalities, market setup costs, the creation of infrastructure, and returns to scale in the production of public goods all create preferences for living in large population centers. Alternatively, the population may enter the utility function negatively. This assumption applies if agents require land, either for agriculture or suburban pleasures.

An agent's utility from a location may also depend upon its average distance to other agents, what I call *separation*.[5] If agents face significant transportation costs, and if they trade with a significant percentage of the other agents, then they may wish to minimize their average distance to other agents. Average distance could also enter into utility positively; agents may wish to be as far from other agents as possible. Supposing for a moment that the agents represent firms. Some firms might impose negative externalities on others (coal plants and laundries come to mind). Preferring to live far from other agents subtly differs from preferring to live with few other agents, but as we shall see the equilibrium distributions they generate differ significantly.

Distance measures can be based on city block or Euclidean metrics. City block distance measures the number of city blocks vertically and horizontally separating two locations. The city block distance between location (1,1) and location (3,3) equals four. Think of walking in a city. A city block path from (1,1) to (3,3) first proceeds two blocks to (1,3) and then from (1,3) two more blocks to (3,3) for a total of four blocks. I denote the city block measure by $d^c(i, j, F)$.

Def'n: *The* **city block measure** *from* (i^*, j^*)*, given* F*,*

$$d^c(i^*, j^*, F) = \frac{1}{m(n-1)^2} \sum_{i=1}^{n} \sum_{j=1}^{n} F_{ij} \cdot (\mid i^* - i \mid + \mid j^* - j \mid)$$

The $m(n-1)^2$ divides the total distance by the number of agents multiplied by the maximal possible distance to create a normalized average. The subsequent analysis includes only the city block measure as it lends itself more readily to formal analysis. Similar results hold if the Euclidean distance measure is used.

2.1 Equilibria

Equilibria distributions of cities depend on the ability of agents to relocate. If agents can move anywhere, the equilibrium distribution of cities will be much different than if they can only move locally. To capture these differences, I consider two rules: *global*

[5]Note that for population preferences the entire population distribution does not matter, only the population at the location. The separation variable relies on information about the entire distribution.

relocations and *local relocations*. Under global relocations, each agent chooses the location on the lattice generating the highest utility. Agents take into account the effect of their own movement on population.[6] In the event of a tie, an agent chooses the first location evaluated from among those generating the highest utility.

Def'n: *The distribution F is an* **equilibrium with respect to global relocations** *if and only if $F_{ij} > 0$ implies $u(i,j,F) \geq u(i',j',F - \delta_{ij} + \delta_{i'j'})$ for all i', j', where $\delta_{ij} : N \times N \to \{0,1\}$ where $\delta_{ij}(i,j) = 1$ and $\delta_{ij}(\hat{i},\hat{j}) = 0$ for $(\hat{i},\hat{j}) \neq (i,j)$.*

Under local relocations, agents possess limited vision preventing them from searching the entire lattice for the best location. Agents move to the location within a fixed neighborhood that generates the greatest utility. Once there, they may become aware of an even better location and move again. However, agents cannot look ahead. An agent could not move from a city with twenty agents to one with five agents because the latter was in the neighborhood of a city with sixty agents.

Despite the fact that locally relocating agents do not know the entire population distribution, they can compute their utility from each location in their local neighborhood. This assumption may appear problematic for preferences that include separation. Utility calculations use the entire population distribution, something the agents do not know. This apparent contradiction is resolved if the population distribution determines economic variables, such as prices, that in turn determine utility. The map from the distribution to the economic variables need not be invertible, so the agents cannot always deduce the population distribution given prices.

Def'n: *The distribution F is an* **equilibrium with respect to local relocations of distance** *d if and only if $F_{ij} > 0$ implies $u(i,j,F) \geq u(i',j',F - \delta_{ij} + \delta_{i'j'})$ for all i', j' such that $(\mid i - i' \mid + \mid j - j' \mid) \leq d$* where $\delta_{ij}(i,j) = 1$ and $\delta_{ij}(\hat{i},\hat{j}) = 0$ for $(\hat{i},\hat{j}) \neq (i,j)$.

In the examples described below the size of a neighborhood equals one ($d = 1$). The city block measure restricts agents' movements to one of the four locations above and below and to the right and left on the lattice. An agent residing at a corner location has only two alternative locations.

Two methodological issues remain: the timing of relocation decisions and the initial population distributions. The timing of updating, whether synchronous or asynchronous, often qualitatively effects both dynamics and end states (Huberman and Glance [12]). Given that relocation decisions might be made at any time, in this model agents do not move simultaneously but instead relocate at different times. The order of the asynchronous updating occurs randomly.[7] Agents can be identified

[6] This contrasts with many Tiebout models that assume that agents do not consider the effect of their own relocations.

[7] An alternative approach would be to use incentive based asynchronous updating. Under incentive based asynchronous updating, utility differences determine the order of agents' relocations. Those agents with the most to gain from relocating, move first. Page [23] shows that incentive based asynchronous updating alters both dynamics and the distribution over end states for several classes

by numbers ranging from 1 to m and placed in a queue. In each period, the agents sequentially choose to reside in the location offering the highest utility given the relocations of all agents ahead of them in the queue. Initially, agents choose locations randomly according to a uniform distribution: equal probability of each location. Prior to industrialization, most countries' economies were agriculturally based. As farmers tend to spread throughout the countryside, the uniformity assumption accords with reality.

3 Linear Preferences

I begin by considering linear preferences over separation and its population. The analysis consists of two parts. I first analyze models where agents' preferences include only one the two variables. In the second part, I consider all possible linear combinations of location population and separation.

3.1 Population Preferences

As mentioned, population may enter agents' utilities either positively or negatively, creating two scenarios that I call agglomeration and isolation. Within each scenario, agents may relocate either globally or locally.

3.1.1 Agglomeration

In the first scenario, an agent's utility equals the population at her home location: an agent residing at location (i, j) obtains a utility equal to F_{ij}. The larger the local population, the more utility accruing to the agent. In the global relocations scenario, the first agent to relocate chooses from among those locations with the largest population. All subsequent agents choose the same location as the first agent. Similar to Arthur [1], this scenario exhibits substantial sensitivity to initial conditions in location, the lattice site initially containing the most agents becomes the large city. But it exhibits no sensitivity in the population distribution (only one city is formed).

The next two claims state that a distribution F is an equilibrium with respect to global relocations if and only if all agents reside at a single location and that these equilibrium distributions maximize utility.

Claim 1 *If $u(i, j, F) = F_{ij}$ then F is an equilibrium with respect to sequential global relocations if and only if there exists an (i, j) such that $F_{ij} = m$*

pf: If $F_{ij} = m$ then the utility from remaining at (i, j) equals m and the utility from any other location equals 1. Thus, F is an equilibrium.

To prove the other direction, let $K = \max_{(i,j)}\{ F_{ij}\}$ where $K < m$ and show that this leads to a contradiction. There are two cases to consider. First,

of cellular automata.

suppose there exists a unique (i^*, j^*) such that $F_{ij} = K$. For all $(i, j) \neq (i^*, j^*)$, $u(i, j, F) < u(i^*, j^*, F)$, implying that all agents must be located at (i^*, j^*), a contradiction. Second, suppose $F_{ij} = F_{i^*j^*} = K$ and that $(i, j) \neq (i^*, j^*)$. Consider an agent located at (i, j). Her utility equals K. Her utility from (i^*, j^*) equals $K + 1$, a contradiction. □

Claim 2 *If $u(i, j, F) = F_{ij}$ then F is utility maximizing if and only if there exists an (i, j) such that $F_{ij} = m$*

pf: Maximizing the utilitarian social welfare function can be shown to be equivalent to the following constrained maximization problem.

$$\max \sum_{i=1}^{n} \sum_{j=1}^{n} F_{ij} \cdot F_{ij}$$

subject to

$$\sum_{i=1}^{n} \sum_{j=1}^{n} F_{ij} = m \text{ and } F_{ij} \geq 0 \text{ for all } (i, j)$$

The convexity of the objective function implies that optimum must occur at a corner. All corners obtain an identical value of m^2. □

In the local relocations scenario, the dynamics become more complicated. Each agent in turn chooses a location in her neighborhood with maximal population. In this and other simulations, I used one thousand agents on a nine by nine lattice. The percentage of agents in each location from a representative run are shown in Figure 1.

Place Figure 1 Here

In the first round of relocations, all agents do not choose to reside at the same location. This occurs for two reasons. First, agents' neighborhoods need not intersect, so some agents cannot possibly choose to reside at the same location. Second, even among those agents whose neighborhoods do intersect, the locations within their neighborhoods with maximal population may differ. After a few rounds of relocations, the population pattern consists of a collection of small villages. For a distribution to be an equilibrium, any two locations with nonzero populations may not be adjacent (see Figure 1).

Under local relocations, the equilibrium population distributions exhibit limited sensitivity to initial conditions both in location and in distribution. By changing the initial locations of a few agents, not only can the location of the small villages change but so can their populations, but the populations and locations of those villages far from the relocated agents usually remain unaffected.

3.1.2 Isolation

As would be expected, cities do not form if agents want to live in lightly populated areas. If $u(i,j,F) = -F_{ij}$, then under global relocations, the agents spread themselves uniformly over the lattice as stated in Claim 3.

Claim 3 *Assume $m = \alpha \cdot n^2$, where α is an integer. If $u(i,j,F) = -F_{ij}$, then F is an equilibrium with respect to global relocations if and only if $F_{ij} = \alpha$ for all (i,j)*

pf: First, $F_{ij} = \alpha$ for all (i,j) is shown to be an equilibrium. Given this distribution, all agents obtain a utility equal to $-\alpha$. The utility to an agent currently located at (i,j) from an alternative location equals $-(\alpha+1)$, thus F is an equilibrium.
The uniqueness of the equilibrium is shown by contradiction. Suppose that there exists an (i,j) such that $F_{ij} \geq (\alpha+1)$. It follows that there exists an (i',j') such that $F_{i'j'} \leq (\alpha-1)$. Any agent located at (i,j) would obtain strictly greater utility by moving to (i',j'). □

The next claim states that the equilibrium under global relocations also maximizes utility.

Claim 4 *Assume $m = \alpha \cdot n^2$, where α is an integer. If $u(i,j,F) = -F_{ij}$ then F is utility maximizing if and only if $F_{ij} = \alpha$ for all (i,j).*

pf: The relevant Lagrangian is

$$\max_{F,\lambda} \sum_{i=1}^{n} \sum_{j=1}^{n} -F_{ij} \cdot F_{ij} \ + \ \lambda \cdot (\sum_{i=1}^{n} \sum_{j=1}^{n} F_{ij} \ - m)$$

The first order necessary conditions:

$$2 \cdot F_{ij} = \lambda \quad \text{for all} \quad (i,j)$$

$$\sum_{i=1}^{n} \sum_{j=1}^{n} F_{ij} = m$$

are sufficient because of the strict concavity of the function. Therefore, the interior critical point describes a unique maximum where all locations have identical populations. □

Under local relocations, the equilibrium distributions resulting from computational experiments appear close to uniform. If the initial distribution deviates from uniform, then the final distribution may differ substantially as shown in Figure 2.

place Figure 2 here

By the previous claim, this distribution does not maximize utility. As for agglomeration preferences, under local relocations, equilibrium distributions exist which do not maximize utility. This occurs because population preferences do not create smooth utilities over the lattice. Utility equals either population or its negation. Therefore, the utility function has multiple local peaks that collect agents of limited search ability.

3.2 Separation Preferences

Separation preferences assume that agents care about their average distance to other agents, and not on the population at their home location. Separation preferences capture economic features like transportation costs and pollution that depend upon distance. For example, if agents trade extensively, they may wish to be as close on average as possible to other agents. Alternatively, if agents do not trade and if other agents create pollution that destroys air, land, and water quality, then agents may wish to distance themselves from other agents.[8]

Separation preferences differ from population preferences in that they create smooth utility functions on the lattice. Given a distribution of agents, the graph of the utilities at each location on the lattice does not contain any non–global, local peaks in equilibrium, implying identical equilibrium sets for agents who rely on global and local relocations. I formalize this intuition in the next claim.

Claim 5 *If $u(i,j,F) = h(d^c(i,j,F))$ where h is a strictly monotonic real valued function then F is an equilibrium with respect to global relocations if and only if it is an equilibrium with respect to local relocations.*

pf: see appendix.

Identical equilibrium sets need not imply similar dynamics under the two scenarios. Therefore, beginning with identical initial distributions, global and local relocations typically do not generate the same equilibrium.

3.2.1 Attraction

I first assume that agents prefer to be close to other agents. Therefore, $u(i,j,F) = -d^c(i,j,F)$. In the global relocation scenario, the first agent to relocate chooses the location with the minimal average distance to all other agents. Given the assumption of a uniform initial distribution of agent locations, this agent locates near the center of the lattice. The location chosen by the next agent and all subsequent agents will be either be the same, or differ by a small distance.[9] After a few rounds of adaptation,

[8] If agents hate visitors and if the probability that an agent visits another agent decreases linearly in distance, then agents would have preferences which increase in separation.

[9] If for example, every agent except for one locates in the center, and the outlying agent resides far to the north, then the first agent to relocate (unless he is the outlier) may move out of the city. If the northerner relocates next, then she will chose to live in the city.

typically two or three, all agents reside in a single location. The end states differ from the agglomeration scenario in that here the resulting single city lies at or near the center of the lattice. There is little sensitivity to initial conditions in location and none in population distribution.

Under local relocations, the convergence is even slower, but again one city forms. In each time period agents march towards the center of the lattice. Eventually, all agents reside at a single location at or near the center of the lattice.

The next two claims state that the unique equilibrium consists of a single city and that this equilibrium maximizes utility.

Claim 6 *If $u(i,j,F) = -d^c(i,j,F)$ then F is an equilibrium with respect to global and local relocations if and only if there exists an (i,j) such that $F_{ij} = m$*

pf: First suppose that there exists an (i,j) such that $F_{ij} = m$. It follows that $u(i,j,F) = 0$. If any agent relocates, her utility would be strictly negative. Therefore, F is an equilibrium. The other direction is proven by contradiction. Suppose that all agents do not reside at one location. Choose (i,j) and (i',j') such that $F_{ij} > 0$, $F_{i'j'} > 0$, and $i < i'$. It suffices to show that an agent can benefit by relocating. Let $s(i) = \sum_{j=1}^{n} F_{ij}$ denote the number of agents in row i on the lattice. Ignoring the constant term in the city block measure, if an agent moves from (i,j) to $(i+1,j)$, her change in utility Δ_i given by

$$\Delta_i = 1 - \sum_{k=1}^{i} s(k) + \sum_{k=i+1}^{n} s(k)$$

Δ_i consists of three terms. The first term, 1, takes into account the fact that the agent does not move one unit away from herself when she relocates. The second term captures the fact that she moves one unit away from all agents in rows 1 through i. The third term captures the fact that she moves one unit closer to all agents in rows $i+1$ to L. Similarly, when an agent moves from (i',j') to $(i'-1,j')$, her change in utility $\Delta_{i'}$ given by

$$\Delta_{i'} = 1 - \sum_{k=i'}^{n} s(k) + \sum_{k=1}^{i'-1} s(k)$$

There are two cases to consider. $i' = i+1$ and $i' > i+1$. If $i' = i+1$ then $\Delta_i + \Delta_{i'} = 2$. If $i' > i+1$, then

$$\Delta_i + \Delta_{i'} = 2 + 2 \cdot \sum_{k=i+1}^{i'-1} s(k)$$

In either case, $\Delta_i + \Delta_{i'} > 0$ implying that one of the terms exceeds zero and completing the proof. □

Claim 7 *If $u(i,j,F) = -d^c(i,j,F)$ then F is utility maximizing if and only if there exists an (i,j) such that $F_{ij} = m$*

pf: $u(i,j,F) \leq 0$ for all F. Therefore, it suffices to show that there exists an (i,j) such that $F_{ij} = m$ if and only if

$$U(F) = \sum_{i=1}^{n}\sum_{j=1}^{n} u(i,j,F)F_{ij} = 0$$

Suppose $F_{ij} = m$, then $F_{i'j'} = 0$ for all $(i',j') \neq (i,j)$. A straightforward calculation shows that $U(F) = 0$. To prove the other direction, suppose that $F_{ij} < m$ for all (i,j). Choose (i,j) and (i',j') so that $F_{ij} > 0$ and $F_{i'j'} > 0$. It follows that $d^c(i,j) > 0$ and $d^c(i',j') > 0$ implying that the $U(F) < 0$. □

The theorem does not say that the equilibria and utility maximizing distributions have an inherent bias towards the center. Yet, in computational experiments, the city always lies near the center because initially, agents move to locations close to other agents. In equilibrium, the central location plays no role. Thus, although this scenario creates outcomes similar to those found by Krugman ([15] and [16]) (both generate single cities near the center with slight sensitivity to initial conditions), they do so for different reasons. In Krugman's models, farmers remain in the surrounding areas and the city lies near the center to be close to agricultural products. In the model presented here, in equilibrium, the city's location is immaterial. It is a historical artifact of the initial distribution and the city formation process. A glimpse at any map reveals that many cities lie on rivers even though currently those rivers have only marginal impacts on the economy.

3.2.2 Repulsion

The alternative assumption that agents wish to *maximize* their average distance to other agents, leads to a counter–intuitive result. Agents end up living in cities. More precisely, they form four cities in the four corners of the lattice with opposite corners having equal populations. (See Figure 3) Here, a macro phenomenon *emerges* in the true sense. The aggregation of population in the corners runs counter to the microlevel incentives to disperse. Cities form even though no one wants cities.

place Figure 3 here

Distributions like those shown in Figure 3 are both equilibria and utility maximizing for agents who want to maximize separation.

Claim 8 *Assume m is even. If $u(i,j,F) = d^c(i,j,F)$, then F is an equilibrium with respect to global relocations if and only if F satisfies the following equalities:*

$$\begin{aligned} F_{11} &= F_{nn} \\ F_{1n} &= F_{n1} \\ F_{ij} &= 0 \qquad \text{if } \{i,j\} \not\subseteq \{1,n\} \end{aligned}$$

pf: see appendix.

Claim 9 *Assume m is even. If $u(i,j,F) = d^c(i,j,F)$, then the distribution defined in Claim 8 is utility maximizing.*

pf: see appendix

Given these preferences, the dynamics differ slightly between the local and global relocation scenarios. Under global relocations, the first agent chooses a corner, as do all other agents in turn. If the populations in opposite corners differ, then agents continue to relocate until attaining equilibrium. In the local relocation scenario, agents crawl towards the corners of the space, and after a few iterations, all agents reside along the edges of the lattice. From there they converge to the corners. Both cases exhibit moderate sensitivity to initial condition in population distribution, but none in location.

This finding can be criticized because both for its reliance on the square lattice and for being unrealistic. The first criticism is moot for two reasons. First, whether looking at cities forming within a country or locational decisions within a city, the relevant topology is two dimensional. Second, I can generate a similar emergence on a sphere by allowing nonlinear preferences as I show in section 4. As for this not being realistic, that point is accepted. I do not intend this to be a realistic description of why cities form. Instead, the result serves a pedagogical purpose. It demonstrates that cities can form without an explicit preference for agglomeration or against separation. Such assumptions, though sufficient, are not necessary for cities. To borrow from Schelling [25], micromotives and macrobehavior need not appear consistent.

3.2.3 Summary

The table below summarizes the four cases with pure separation and agglomeration.[10] Notice that some of the scenarios exhibit extreme sensitivity to initial conditions in the location of cities, others exhibit moderate sensitivity in the distribution of city sizes but not in city location.

Summary of Single Component Models			
$u(i,j,F)$	*Global Relocations*	*Local Relocations*	*Utility Max*
F_{ij}	one city anywhere	isolated villages	one city anywhere
$-F_{ij}$	uniformly spread	uniformly spread	uniformly spread
d_{ij}	four corners	four corners	four corners
$-d_{ij}$	one city near center	one city near center	one city anywhere

[10]In the case where agents relocate locally and wish to minimize population, there exist nonuniform equilibrium distributions, but the typical simulation yields a uniform, or near uniform distribution.

3.3 Separation and Agglomeration

I now include both variables in the agents' utility functions but retain the linearity assumption. For expediency, I employ a genetic algorithm to search for optimal population distributions rather than attempting to prove theorems (Holland [11], Goldberg [8]).[11] The coefficients of population and distance may be either positive or negative, resulting in four cases. I assume that the coefficient on the agglomeration term is either plus or minus one and that the coefficient on the separation term is plus or minus α, where α is a positive real number greater than zero.

3.3.1 Isolated Attraction ($-F - \alpha d^c$)

With negative coefficients on both population and separation, agents prefer lightly populated locations, spatially near other agents. The variable $\alpha > 0$ measures the relative weight on the distance component. With either global or local relocations, the equilibria resemble piles of sand near the center of the lattice, in which agents balance their desire to be nearer other agents with their preference for a less crowded home location. (See Figure 4) In hundreds of simulations, the sand pile always appeared (up to translations and rotations). Hence, there is no sensitivity to initial conditions in either location of population distribution. The sand pile has many of the features of the original central place models of Christaller [4]. The city in the center has the largest population, and city population decreases with distance to the center. The core assumptions of central place theory, that farmers want to be close to markets but live in isolation, are consistent with these preferences.

place Figure 4 here

Changes in the relative weight of the two components changes the equilibrium distributions in intuitive directions. As α increases (decreases) the piles of sand grow taller (shorter) and encompasses a smaller (larger) area. Larger α imply that the agents want to be closer together, thereby increasing the population at the center. The sand pile formation appears to vary smoothly with changes in α. A genetic algorithm searching the space of distributions discovered the sand pile formation and was unable to find another distribution generating higher utility.

3.3.2 Agglomerated Attraction ($F - \alpha d^c$)

Switching the sign on the coefficient of F_{ij} creates agents who prefer to live in large cities and close to other agents. Under global relocations, a single city forms near the center of the lattice. The separation component makes the location of the city less sensitive to initial conditions than in the case where utility depended only on population. The proof that this distribution maximizes utility follows from Claim 2

[11] The genetic algorithm I employed uses tournament selection and uniform crossover to search for utility maximizing population distributions.

and Claim 7. With local relocations, agents can become stuck in moderately sized, spatially segregated cities. The findings vary depending upon the relative weights on population and separation. If the population term predominates, then the equilibrium distributions may be similar to those shown in Figure 5, while if the coefficient of the separation term predominates, then the agents move to a single city in the center.

place Figure 5 here

3.3.3 Agglomerated Repulsion ($F + \alpha d^c$)

With positive coefficients on both population and separation, agents prefer to live in cities but far from other people. These are somewhat contradictory preferences, and as might be expected, the end states vary significantly under local and global relocation. Under global relocation, the agents move to two opposing corners in the first iteration. Typically, one of these corners has a larger population than the other. In the second generation, all agents move to the corner with the larger population. This occurs because the population term begins to predominate. Therefore, we see some sensitivity to initial conditions in location, but none in population. It can be shown that the single city in the corner maximizes utility for $\alpha < 2$. Under local relocation, the agents move towards the corners. Within a few generations, the population spreads unevenly over the four corner locations. When the population term predominates, then in addition to the four corner location, there may also be locations just off center with positive population as shown in Figure 6.

place Figure 6 here

3.3.4 Isolated Repulsion ($-F + \alpha d^c$)

In the final scenario, agents prefer locations with small populations that are far from other agents on average. Under global relocation, the agents reside along the edges of the lattice, with a concentration of agents near the corners (See Figure 7).

place Figure 7 here

This end state appears to occur regardless of the initial distribution, so their is little or no sensitivity to initial conditions. Under local relocation, the agents move towards the edges and often converge to the same equilibrium distribution as under global relocation. Sometimes the end states differ slightly as asymmetries cannot be overcome by local movements. Computations using a genetic algorithm suggest that these configurations also maximize utility. The following table summarizes the four two component linear cases.[12]

[12] By utility maximizing I mean the best that could be found by a genetic algorithm

Summary of Linear Models			
Preferences	*Global*	*Local*	*"Utility Maximizing"*
$F_{ij} + d_{ij}$	one city in corner	cities in four corners	one city
$F_{ij} - d_{ij}$	one city in center	several cities near center	one city
$-F_{ij} + d_{ij}$	edges with corner peaks	edges with corner peaks	edges with corner peaks
$-F_{ij} - d_{ij}$	central sand pile	central sand pile	central sand pile

Note the substantial additivity in the equilibrium configurations. Attraction leads to a city in the center, and isolation leads to a uniform distribution. Combining the two (isolated attraction) yields a centrally located sand pile. Checking other pairs of effects reveals a common pattern. No externalities between the two effects reveal themselves in the equilibrium distributions.

4 Nonlinear Preferences

I now extend the model to include nonlinear effects. I consider two cases: one with a nonlinear population term demonstrating that equilibria need not be efficient and that efficient equilibria need not be attained, and one with a nonlinear separation term exhibiting extreme sensitivity in both the location and size distribution of cities to initial conditions.

4.1 Crowding Effects

The first scenario assumes a negative second order effect on population. It replicates well known results within the literature, that if the negative externality from agglomeration is not priced, cities will grow too large. Standard explanations for negative external effect include congestion, pollution, crime, or inefficiencies in public good provision. The utility to an agent from being in location (i, j) can be written as

$$u(i, j, F) = F_{ij} - \alpha \cdot (F_{ij})^2$$

where α is a positive constant. In the global relocation scenario, the dynamics begin similar to the pure agglomeration case. The first agent chooses from among the locations with the largest population, and the second agent chooses the same location. At some point the city becomes overcrowded, and an agent chooses the location with the second largest population. The result at the end of one round of relocations will be several cities, the exact number varies directly with α. All but the last of these cities to form will have identical populations. In the second round of relocations, agents move from the larger cities to the one smaller city until all cities have equal populations. The cities that form tend to be just shy of twice their optimal size. These oversized cities generate utility only marginally higher than the initial uniform distribution.

The utility maximizing size of the cities is $\frac{1}{2\alpha}$. And distributions consisting of only cities of size $\frac{1}{2\alpha}$ are locally stable equilibria as well.[13]

That crowding effects can create non optimal equilibria is not a new idea (Mills [21]). What merits mentioning is that although there exist utility maximizing, locally stable equilibria (cities of size $\frac{1}{2\alpha}$) these equilibria have small basins of attraction, i.e. few initial distributions lead to them. Thus, mathematical proofs of the existence of efficient, stable equilibria should be viewed skeptically, unless these equilibria have also been shown to be unique or to have large basins of attraction.

4.2 Partial Separation

Finally, assume that agents like separation but that they do not want to be too far from the other agents. These preferences can be written as follows.

$$u(i, j, F) = d^c_{ij} - \alpha \cdot (d^c_{ij})^2$$

For small values of α, cities form near the corners, just as in the pure separation case. There are multiple equilibria in this case. The proof of this hinges on their being a separation $d^{c*} = \frac{1}{2\alpha}$, that maximizes utility. Any population distribution where every agent is separated from the others by $\frac{1}{2\alpha}$ is utility maximizing, and an equilibrium. Furthermore, it can be shown that these are the only equilibria given these preferences provided the lattice is large enough and has enough sites (i.e. n must be large).[14] The set of distributions where all agents have identical separation is huge. It includes a large set of horizontally and vertically symmetric distributions. For example, in addition to the distribution shown in Figure 8, a distribution consisting of two cities separated by $2d^{c*}$ lattice sites will generate identical separation for all agents.

Place Figure 8 Here

This scenario merits attention for two other reasons. First, once again, cities emerge. Agents prefer neither agglomeration nor distance minimization. Second, it exhibits extreme sensitivity of both the location and the population distribution of cities to initial conditions. Changing the location of one agent can lead to very different equilibria, though of course both equilibria generate the same utility. Interestingly, I did not find extreme sensitivity to initial conditions in all cases despite the enormous number of equilibria.

[13] By locally stable, I mean that if you randomly relocate a few agents, subsequent relocations will return to these equilibria.

[14] The proof is straightforward. Any agent not the ideal distance from other agents can always relocate so she is.

5 Discussion

In this paper, I have constructed an agent location model where agents' preferences depend upon a location's population and its distance to other agents. Within this simple framework, I examined the formation and emergence of cities, their equilibrium spatial distributions, the optimality of these equilibrium distributions, the sensitivity of equilibria to initial conditions, and the differences between global and local relocation scenarios. The model does not include economic variables explicitly, though economic forces underpin the behavior captured by separation and population. By considering a variety of preferences in this framework, I captured some of the behavior described in more sophisticated models: Sensitivity in location to initial conditions results from preferences for population; Central placement appears to be the residue of preferences to minimize distances to other agents; And central places with decreasing population gradients result from preferences favoring proximity to others but local isolation. I also generated some new insights: Many equilibria can be shown to be efficient; Preferences for greater separation lead to emergent cities; With crowding effects, efficient, stable equilibria may exist but the stable, inefficient equilibria tend to emerge from a uniform starting distribution; and nonlinearities in preferences for distance can create emergent cities whose size and location are both extremely sensitive to the initial population distribution.

The two variable model described in this paper can be extended to include natural advantage, heterogenous agents, and population growth. Natural advantages requires giving agents higher utility from residing at or near certain locations. In preliminary simulations in the pure agglomeration model, natural advantage tends to reduce the sensitivity to initial conditions. Heterogeneous preferences reveal surprising additivity across equilibrium distributions. Combinations of two microlevel preferences tend to generate convex combinations of their individual equilibrium distributions. Heterogeneity also makes emergent cities more likely. It is straightforward to construct models with k types of agents, each with different nonlinear preferences over separation, where the equilibrium distribution consists of k cities. Finally, allowing population growth can cause novel dynamics. In the case of negative second order agglomeration effects, cascades occur. New cities form and spur relocations by many agents.

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Figure 1: (local)

$$u(i, j, F) = F_{ij}$$

								8
	7							
				12			7	
								7
	6				11			
		8						
					9		12	
	7							
						6		

Figure 2: (local)

$$u(i, j, F) = -F_{ij}$$

				1				
		1	1	2	1	1		
	1	1	2	3	2	1	1	
1	1	2	3	4	3	2	1	1
1	2	3	4	5	4	3	2	1
1	1	2	3	4	3	2	1	1
	1	1	2	3	2	1	1	
		1	1	2	1	1		
				1				

Figure 3:

$$u(i, j, F) = d^c(i, j, F)$$

18								32
32								18

Figure 4:

$$u(i, j, F) = -F_{ij} - \alpha d^c(i, j, F)$$

				3	3			
			4	8	8	4		
		1	7	11	11	7	1	
			4	8	8	4		
				3	3			

Figure 5: (local)

$$u(i, j, F) = +F_{ij} - d^c(i, j, F)$$

			16					
						41		
				27				
						16		

Figure 6: (local)

$$u(i, j, F) = F_{ij} + d^c(i, j, F)$$

19								23
	7							
	6							
21								24

Figure 7:

$$u(i,j,F) = -F_{ij} + d^c(i,j,F)$$

6.2	3.7	2.5	1.8	1.6	1.8	2.5	3.7	6.2
3.7	1.2						1.2	3.7
2.5								2.5
1.8								1.8
1.6								1.6
1.8								1.8
2.5								2.5
3.7	1.2						1.2	3.7
6.2	3.7	2.5	1.8	1.6	1.8	2.5	3.7	6.2

Figure 8:

$$u(i,j,F) = d^c(i,j,F) - \alpha(d^c(i,j,F))^2$$

		9.5			15.5			
	3.3					5		
9.3							7.4	
7.4							9.3	
	5					3.3		
		15.5			9.5			

Appendix

Claim 4 *If* $u(i,j,F) = h(d^c(i,j,F))$ *where* h *is a strictly monotonic real valued function then* F *is an equilibrium with respect to global relocations if and only if it is an equilibrium with respect to local relocations.*

pf. The only if direction holds by definition. Therefore, it suffices to show that if a distribution F is an equilibrium with respect to local relocations, then it is also an equilibrium with respect to global relocations when $u(i,j,F) = h(d^c(i,j,F))$ and h is monotonic. Without loss of generality, assume that h is monotonically increasing. It suffices to show that the result holds for the case where h is the identity function.

The proof proceeds by contradiction. Suppose that an agent residing at (i,j) would benefit by moving to (i',j'), where $i' > i$ and $j' \geq j$. The utility increase in moving from (i,j) to (i',j') is the sum of the increase of moving from (i,j) to (i',j) and the increase in moving from (i,j) to (i,j'). Therefore, let Δ_i equal the change in utility if the agent moves from (i,j) to (i',j) (suppressing the constant term $\frac{1}{m(n-1)^2}$ It is sufficient to prove that if this is strictly positive then Δ_1, the change in utility if the agent moves from (i,j) to $(i+1,j)$ is also strictly positive. Let $s(i) = \sum_{j=1}^{n} F_{ij}$. It is straightforward to show that

$$\Delta_1 = 1 - \sum_{k=i+1}^{n} s(k) + \sum_{k=1}^{i} s(k)$$

If $i' - i = 1$ then $\Delta_i = \Delta_1$. Assume $i' - i \geq 2$, it follows that

$$\Delta_i = (i'-i) + \sum_{k=i'}^{n}(i'-i)\cdot s(k) + \sum_{k=1}^{i}(i'-i)\cdot s(k) + \sum_{k=i+1}^{i'-1}((i'-i) - 2(k-i))\cdot s(k)$$

Multiplying Δ_1 by $(i'-i)$ and subtracting Δ_i yields

$$(i'-1)\cdot\Delta_1 \ - \ \Delta_i \ = \ \sum_{k=i+1}^{i'-1} 2(k-i)s(k)$$

Since all of the $s(k)$'s are greater than or equal to zero, it follows that $\Delta_1 > 0$. □

Claim 6: *Assume* m *is even. If* $u(i,j,F) = d^c(i,j,F)$*, then* F *is an equilibrium with respect to global relocations if and only if* F *satisfies the following equalities:*

$$\begin{aligned} F_{11} &= F_{nn} \\ F_{1n} &= F_{n1} \\ F_{ij} &= 0 \qquad \textit{if } \{i,j\} \not\subseteq \{1,n\} \end{aligned}$$

pf: The proof proceeds in two parts. First, if $\{i,j\} \not\subseteq \{1,n\}$ then it is shown that $F_{ij} = 0$. Then, the equality of the populations in opposite corners is shown.

Part 1: Suppose that an agent resides at (i,j) and $\{i,j\} \not\subseteq \{1,n\}$. Without loss of generality assume that $i \not\in \{1,n\}$. Let $s(i) = \sum_{j=1}^{n} F_{ij}$. Let Δ_+ equal the change in utility if the agent moves to $(i+1, j)$ and Δ_- equal the change in utility if the agent moves to $(i-1, j)$, again without the constant term. It is straightforward to show that

$$\Delta_+ = 1 - \sum_{k=i+1}^{n} s(k) + \sum_{k=1}^{i} s(k)$$

and that

$$\Delta_- = 1 + \sum_{k=i}^{n} s(k) - \sum_{k=1}^{i-1} s(k)$$

adding the two terms obtains $\Delta_+ + \Delta_- = 2 + 2 \cdot s(i)$. Since the sum of the two terms is strictly positive, one of the two terms must be positive, completing the first part of the proof.

Part 2: A straightforward calculation shows that if $F_{11} = F_{nn}$ and $F_{1n} = F_{n1}$ then no agent increases her utility by relocating. By symmetry it suffices to show that if $F_{ij} = 0$ for all $\{i, j\} \not\subseteq \{1, n\}$ and if $F_{11} < F_{nn}$ then an agent would relocate from (n, n) to $(1, 1)$. Given these conditions, it follows that

$$u(1, 1, F) - u(n, n, F) = 2(n-1) \cdot [F_{nn} - F_{11}] > 0$$

which completes the proof. □

Claim 7 *Assume m is even. If $u(i, j, F) = d^c(i, j, F)$, then F is utility maximizing if and only if F satisfies the following equalities:*

$$\begin{array}{lcll} F_{11} & = & F_{nn} & \\ F_{1n} & = & F_{n1} & \\ F_{ij} & = & 0 & \text{if } \{i,j\} \not\subseteq \{1,n\} \end{array}$$

pf: The Lagrangian for this problem is as follows:

$$\max_{F,\lambda} 2 \cdot \sum_{i=1}^{n-1} \sum_{i'>i} \sum_{j=1}^{n} \sum_{j'=1}^{n} (i'-i) \cdot F_{ij} \cdot F_{i'j'} + 2 \cdot \sum_{j=1}^{n-1} \sum_{j'>j} \sum_{i=1}^{n} \sum_{i'=1}^{n} (j'-j) \cdot F_{ij} \cdot F_{i'j'} + \lambda \cdot (\sum_{i=1}^{n} \sum_{j=1}^{n} F_{ij} - m)$$

Let $s(i) = \sum_{j=1}^{n} F_{ij}$ and $r(j) = \sum_{i=1}^{n} F_{ij}$. The first order necessary conditions can be written as follows:

$$\sum_{i=1}^{n} \sum_{j=1}^{n} F_{ij} = m$$

$$\sum_{i'=1}^{n} \mid i' - i \mid s(i') \; + \; \sum_{j'=1^n} \mid j' - j \mid r(j') \; = \; \lambda \; \text{ for all } (i, j)$$

The crux of the proof is that $s(i) = 0$ for $i \notin \{1, n\}$ and that $r(j) = 0$ for $j \notin \{0, 1\}$. Holding j fixed and subtracting the first order necessary condition for $(n-1, j)$ from the first order necessary condition for (n, j) obtains:

$$-s(1) + \sum_{i=1}^{n} s(i) = 0$$

Similarly, subtracting the first order necessary condition for $(2, j)$ from the first order necessary condition for $(1, j)$ obtains:

$$-s(n) + \sum_{i=1}^{n-1} s(i) = 0$$

Summing these two equalities yields

$$\sum_{i=2}^{n-1} s(i) = 0$$

implying that $s(i) = 0$ for $i \in \{2, .., n-1\}$. Substituting into the previous equation, gives that $s(1) = s(n)$. A similar argument shows that $r(i) = 0$ for $i \in \{2, ., n-1\}$ and that $r(1) = r(n)$. A straightforward calculation shows that if $s(1) = s(n) = \frac{m}{2} = r(1) = r(n)$ then all of the first order conditions are satisfied. It also follows from the definitions of s and r that $F_{ij} = 0$ if $\{i, j\} \not\subseteq \{1, n\}$. Therefore, $s(1) = s(n)$ can be rewritten as $F_{11} + F_{1n} = F_{n1} + F_{nn}$ and $r(1) = r(n)$ as $F_{11} + F_{n1} = F_{1n} + Fnn$. Adding the first equation to the second yields $F_{11} = F_{nn}$. Plugging this into either equation yields $F_{1n} = F_{n1}$. A straightforward calculation shows that all distributions satisfying $F_{11} = F_{nn}$ and $F_{1n} = F_{n1}$ have identical values under the utilitarian social welfare function.

It remains to be shown that the first order necessary conditions are sufficient. As a first step in showing sufficiency, it is proven that any distribution F with $F(i, j) > 0$ for some $i \notin \{1, n\}$ has a lower value under the utilitarian social welfare function than a distribution where an agent at location (i, j) moves to either $(i-1, j)$ or $(i+1, j)$. Let Δ_+ equal the change in the sum of the agents' utilities if the agent moves to $(i+1, j)$ and let Δ_- equal the change in the sum of the agents' utilities if the agent moves to $(i-1, j)$. It is straightforward to show that

$$\Delta_+ = 2 \cdot [1 - \sum_{k=i+1}^{n} s(k) + \sum_{k=1}^{i} s(k)]$$

and that

$$\Delta_- = 2 \cdot [1 + \sum_{k=i}^{n} s(k) - \sum_{k=1}^{i-1} s(k)]$$

adding the two terms gives $\Delta_+ + \Delta_- = 4 + 4 \cdot s(i) > 0$. By symmetry, at the global optimum all agents must be located in the four corners. To complete the proof, the population must be shown to have equal populations in the opposite corners. The proof proceeds by contradiction. Suppose that $F_{11} < F_{nn}$. There are two cases to consider.

Case 1: ($F_{nn} \geq F_{11} + 2$) If an agent at location $(1, 1)$ moves to location (n, n) then the change in aggregate utility, Δ is given by:

$$\Delta = 4n[F_{nn} - 1 - F_{11}] > 0$$

Case 2: ($F_{nn} - F_{11} = 1$) By assumption m, the total number of agents, is even; therefore, $F_{1n} \neq F_{n1}$. Without loss of generality assume that $F_{1n} > F_{n1}$. If an agent at location (n, n) moves to location $(n, 1)$, then the change in aggregate utility, Δ, is given by:

$$\Delta = 2n[F_{nn} - 1 - F_{11} + F_{1n} - F_{n1}] > 0$$

which completes the proof. □