Particle hopping models and traffic flow theory

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This paper shows how particle hopping models fit into the context of traffic flow theory, that is, it shows connections between fluid-dynamical traffic flow models, which derive from the Navier-Stokes-equations, and particle hopping models. In some cases, these connections are exact and have long been established, but have never been viewed in the context of traffic theory. In other cases, critical behavior of traffic jam clusters can be compared to instabilities in the partial differential equations. Finally, it is shown how all this leads to a consistent picture of traffic jam dynamics. — In consequence, this paper starts building a foundation of a comprehensive *dynamic* traffic theory, where strengths and weaknesses of different models (fluid-dynamical, car-following, particle hopping) can be compared, and thus allowing to *systematically* chose the appropriate model for a given question.

I. INTRODUCTION

Traffic jams have always been annoying. At least in the industrialized countries, the standard reaction has been to expand the transportation infrastructure to match demand. In this phase of fast growth, relatively rough planning tools were sufficient. However, in the last years most industrialized societies started to see the limits of such growth. In densely populated areas, there is only limited space available for extensions of the transportation system; and we face increasing pollution and growing accident frequencies as the downsides of mobility. In consequence, planning is now turning to a fine-tuning of the existing systems, without major extensions of facilities. This is for example reflected in the United States by the Clean Air Act and by the ISTEA (Intermodal Surface Transportation and Efficiency Act) legislation. The former sets standards of air quality for urban areas, whereas the latter forces planning authorities to evaluate land use policies, intermodal connectivity, and enhanced transit service when planning transportation.

In consequence, planning and prediction tools with a much higher reliability than in the past are necessary. Due to the high complexity of the problems, analytical approaches are infeasible. Current approaches are simulation-based (e.g. [1–4]), which is driven by necessity, but largely enhanced by the widespread availability of computing power nowadays. Yet, also for computers one needs good simplified models of the phenomena of interest: Just coding a perfect representation of reality into the computer is not possible because of limits of knowledge, limits of human resources for coding all these details, and because of limits of computational resources.

Practical simulation has to observe trade-offs between *resolution*, *fidelity*, and *scale* [5]. Resolution refers to the smallest entities (objects, particles, processes) resolved in a simulation, whereas fidelity means the degree of realism in modeling each of these entities, and scale means the (spatial, temporal, ...) size of the problem. It is empirically well known, for example from fluid dynamics, that to a certain extent a low fidelity high resolution model (lattice gas automata [6,7]) can do as well as a high fidelity low resolution model (discretization of the Navier-Stokes-equations), or in short: Resolution can replace fidelity.

Current state-of-the-art traffic modeling has a fixed unit of (minimal) resolution, and that is the individual traveler. Since one is aiming for rather large scales (for example the Los Angeles area consists of approx. 10 million potential travelers), it is rather obvious that one has to sacrifice fidelity to achieve reasonable computing times.

One important part of transportation modeling is road traffic. For example in Germany, road traffic currently contributes more than 81% of all passenger and 52.7% of all freight transportation [8]. And despite widespread efforts, the share of road transportation is still increasing. For that reason, it makes sense to start with road traffic when dealing with transportation systems.

Putting these arguments together, one thing which is needed for large scale transportation simulations is a *minimal* representation of road traffic. Particle hopping models clearly are candidates for this, and even if not, building a *minimal* theory of road traffic is certainly the right starting point.

This paper shows how particle hopping models fit into the context of traffic flow theory. It starts out with a historical overview of traffic flow theory (Section II), followed by a systematic review of fluid-dynamical models for traffic flow (Section III) starting from the Navier-Stokes-equations. Section IV defines different particle hopping models which are of interest in the context of traffic flow. Section V then shows the different connections between the fluid-dynamical traffic flow models and particle hopping models. In some cases, these connections are exact and have long been established, but have never been viewed in the context of traffic theory. In other cases, critical behavior of traffic jam clusters can be compared to instabilities in the partial differential equations. Finally, it is shown how this leads to a consistent picture of traffic jam dynamics (Section VI). A discussion of the consequences for traffic simulations (Section VII) serves as summary and discussion, and a collection of open questions (Section VIII) conclude the paper.

II. HISTORICAL OVERVIEW OF TRAFFIC THEORY

Vehicular traffic has been a widely and thoroughly researched area in the 1950s and 60s. For a review of traffic theory, see, for example, one of [9–11].

Vehicular traffic theory can be broadly separated into three branches: Traffic *flow* theory, *car-following* theory, and one more recent addition: particle hopping models.

A. Traffic *flow* theory

Traffic flow theory is concerned with finding relations between the three fundamental variables of traffic flow, which are velocity v, density ρ, and current or throughput or flow j. Only two of these variables are independent since they are related through $j = \rho\, v$. Possible units for these variables are $[v] = km/h$, $[\rho] = vehicles/km$, and $[j] = vehicles/hour$.

The first approach of traffic flow theory historically was to search for time-*in*dependent relations between j, ρ and v. These relations are the so-called fundamental diagrams. The form of such a relation is, though, still debated in the traffic flow literature [12,13]. The problem stems mainly from the fact that reality measurements are done in non-stationary conditions. There, only short time averages make sense, and they usually show large fluctuations. I will at the end of the paper discuss how a dynamic, particle based description of traffic can resolve these difficulties.

The second step of traffic flow theory was to introduce a dynamic, i.e. time-dependent description. This was achieved by a well-known paper from Lighthill and Whitham, published in 1955 [14]. This paper introduced a description based on the equation of continuity, together with the assumption that flow (or velocity) depend on the density only, i.e. there is no relaxation time, velocity adapts *instantaneously* to the surrounding density.

Prigogine, Herman, and coworkers developed, in the 1960s, a kinetic theory for traffic flow [15]. They derived the Lighthill-Whitham situation as a limiting case of the kinetic theory. Kinetic theory anticipates many of the phenomena (such as start-stop-waves) which come up in later work, but probably because the mathematics of working in this framework is fairly laborious, this theory has not been developed any further until recently [16,17].

Instead, in 1971, Payne replaced the assumption of instantaneous adaption in the Lighthill-Whitham theory by an equation for inertia, which is similar to a Navier-Stokes-equation [18]. Kühne, in 1984, added a viscosity term and initiated using the methods of nonlinear dynamics for analyzing the equations [19–22].

In a parallel development, Musha and Higuchi proposed the noisy Burgers equation as a model for traffic and backed that up by measurements of the power spectrum of traffic count data [23].

In Section III, these fluid-dynamical models will be put into a common perspective.

B. Car-following theory

Car-following theory regards traffic from a more microscopic point of view: The behavior of each vehicle is modeled in relation to the vehicle ahead. As the definition indicates, this theory concentrates on single lane situations where a driver reacts to the movements of the vehicle ahead of him. Many car-following models are of the form

$$a(t+T) \propto \frac{v(t)^m}{[\Delta x(t)]^l} \cdot \Delta v(t) \ , \tag{1}$$

where a and v are the acceleration and velocity, respectively, of the car under consideration, Δx is the distance to the car ahead, Δv is the velocity difference to that car, and m and l are constants. T is a delay time between stimulus and response, which summarizes all delay effects such as human reaction time or time the car mechanics needs to react to input.

Other examples for car-following equations are $v(t+T) \propto \Delta x$ [24,25] or $a(t) \propto V[\Delta x(t)] - v(t)]$ [26,27], where $V[\Delta x]$ gives a preferred velocity as a function of distance headway. See also [28–30].

Mathematically, parts of this theory are very similar to the treatment of atomic movements in crystals, and give results about the stability of chains of cars ("platoons") in follow-the-leader situations.

One of the achievements of traffic theory of this period was that relations between car-following models and *static* flow-density-relations were derived.

Car-following theory will not be treated any further in this paper.

C. Particle hopping models

A more recent addition to the development of vehicular traffic flow theory are particle hopping models. In particle hopping models, a road is represented as a string of cells, which are either empty, or occupied by exactly one particle. Movement takes place by hopping between cells. If all particles are updated simultaneously (parallel update, see below), then the particle hopping model treated in this paper formally are also cellular automata (CA).

The technical difference between car-following and CA models for traffic flow is that in the latter, space and time are discrete, whereas in the mathematical treatment of car-following models, they are continuous. *Simulations* of car-following models (e.g. [26–29,108]) discretize time but use continuous space.

Actually, the first proposition of a CA model for traffic is from Gerlough in 1956 [31] and has been further extended by Cremer and coworkers [32,33]. They implemented fairly sophisticated driving rules and also used single-bit-coding with the goal to make the simulation fast enough to be useful for real-time traffic applications. The bit-coded implementation, though, made it too impractical for many traffic applications.

In 1992, CA models for traffic were brought into the statistical physics community. Biham and coworkers used a model with maximum velocity one for one- and for two-dimensional traffic [34]. One-dimensional here refers to roads etc., and includes multi-lane traffic. Two-dimensional traffic in the CA context usually means traffic on a 2-d grid, as a model for traffic in urban areas. Nagel and Schreckenberg introduced a model with maximum velocity $v_{max} = 5$ for one-dimensional traffic, which compared favorably with real world data [35]. Both approaches were further analyzed and extended in a series of subsequent papers, both for the one-dimensional [28,36–62] (see also [63]) and the two-dimensional (see, e.g., [64–66]) investigations.

That work had two motivations at that time: The primary motivation was again computational speed, but this time to make Monte Carlo analysis possible. The second motivation was to keep the models simple enough to allow analytical treatment. An additional third motivation was added more recently: CA-methodology is planned to be used as a high-speed option in traffic projects in Germany [2] and in the United States [1].

From a theoretical point of view, the methodology of particle hopping models lies between fluid-dynamical and car-following theories and helps to clarify the connections between these approaches. One contribution of this paper is to further improve upon the current understanding and to clarify the relations between particle-hopping models and fluid-dynamical models for traffic flow.

III. FLUID-DYNAMICAL MODELS FOR TRAFFIC FLOW

This section reviews fluid-dynamical models for traffic flow. The models can broadly be distinguished by whether they consider the effects of inertia. Models without considering inertia can be derived from the equation of continuity when velocity or current are considered as functions of the density only. Models considering inertia formally are Navier-Stokes-equations, with a car-specific force term which takes into account that drivers want to drive at a certain desired speed. If the time constant of this force term is set to zero, i.e. assuming *instantaneous adaption* to the surrounding density, the models revert to the non-inertia case.

A. General equations

Papers on traffic flow theory usually start with stating the equations under consideration, without setting them in perspective. I will therefore in this paper attempt a more fundamental approach, similar to conventional fluid-dynamics. The precise presentation of most of these equations is necessary anyhow because the particle-hopping models presented later relate to these equations.

One might use the standard fluid-dynamical conservation equations for mass and momentum as a starting point for a fluid-dynamical description of traffic:

$$\partial_t \rho + \partial_x(\rho\, v) = 0 \qquad \text{(equation of continuity)} \tag{2}$$

and

$$\frac{\mathrm{d}v}{\mathrm{d}t} \equiv \partial_t v + v \cdot \partial_x v = F/m \qquad \text{(momentum equation)}\ , \tag{3}$$

where ρ is the density and v the velocity. $\mathrm{d}/\mathrm{d}t$ is the individual (Lagrangian) derivative, F is the force acting on mass m. Eq. 2 describes mass conservation; Eq. 3 describes the fact that the momentum of a point of mass may only be changed by a force. Obviously, for traffic, F has to include vehicle and driving dynamics.

B. Fluctuations

A standard first step in fluid-dynamics [67] is to assume that v and ρ fluctuate statistically around average values $\langle v\rangle$ and $\langle\rho\rangle$, i.e.

$$v = \langle v\rangle + v'\ , \qquad \langle v'\rangle = 0 \tag{4}$$

and

$$\rho = \langle\rho\rangle + \rho'\ , \qquad \langle\rho'\rangle = 0\ . \tag{5}$$

In this case, one only assumes that $\langle v\rangle$ and $\langle\rho\rangle$ fluctuate *slowly* in space and time; for the general subtleties of hydrodynamical theory see, e.g., [68]. Inserting these relations into (2) and (3) and subsequent averaging over the whole equations (e.g. $\langle\partial_x[(\langle\rho\rangle + \rho')(\langle v\rangle + v')]\rangle = \partial_x\langle\rho\rangle\langle v\rangle + \partial_x\langle\rho' v'\rangle$) yields

$$\partial_t\langle\rho\rangle + \partial_x\langle\rho\rangle\langle v\rangle + \partial_x\langle\rho' v'\rangle = 0 \tag{6}$$

and

$$\partial_t\langle v\rangle_L + \langle v\rangle_L \partial_x\langle v\rangle_L + \frac{1}{2}\partial_x\langle v' v'\rangle = \langle F/m\rangle\ . \tag{7}$$

One often parameterizes averaged fluctuations by the corresponding gradient (see, e.g., [67]) $\langle v' A'\rangle \approx -\alpha\, \partial_x\langle A\rangle$, [1] which leads to the set of equations

$$\begin{aligned} \partial_t \rho + \partial_x(\rho\, v) &= D\, \partial_x^2 \rho \\ \partial_t v + v\, \partial_x v &= \nu\, \partial_x^2 v + F/m\ , \end{aligned} \tag{8}$$

where, according to convention, the averaging brackets have been omitted, and the diffusion coefficient D as well as the (kinematic) viscosity ν are assumed to be independent of x and t. It should be noted that similar diffusion terms can also be obtained from other arguments.

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[1] The idea behind this parametrization is that, if there is more than average of quantity A (i.e. $A > \langle A\rangle$ or $A' \equiv A - \langle A\rangle > 0$) at one location, and less than average of quantity A at a neighboring location, then velocity fluctuations represented by v' tend to equilibrate this, and that this happens, to first order, linearly in the concentration gradient of A. (Think of A as, say, red color.)

C. Lighthill-Whitham-theory and kinematic waves

If one assumes that the velocity is a function of density only ($v = f(\rho)$), then the momentum equation is no longer necessary. This corresponds to instantaneous adaption; the particles (or cars) carry no memory. Using without loss of generality the current $j(\rho) \equiv \rho\, v(\rho)$, and setting in addition $D = 0$, from (8) one obtains

$$\partial_t \rho + j'(\rho)\, \partial_x \rho = 0 \tag{9}$$

(Lighthill-Whitham-equation [14]), where $j' = \mathrm{d}j/\mathrm{d}\rho$. For a review of this theory, see, e.g., [14,69].

The equation can be solved by the ansatz $\rho(x,t) = \rho(x - ct)$ with

$$c = j'(\rho)\ . \tag{10}$$

This allows the solution of the characteristics (see, e.g., [69]): A region with density ρ travels with constant velocity $c = j'(\rho)$, and the resulting straight line in space-time is called characteristic. When $j(\rho)$ is convex, i.e. $j'' < 0$, then for regions of decreasing density ($\rho(x_1) > \rho(x_2)$ for $x_1 < x_2$) the characteristics separate from each other. In regions of *in*creasing density, the characteristics come closer and closer together. When two characteristics touch each other, a density discontinuity appears at this place (a front), which moves with velocity

$$c = \frac{j(x_2) - j(x_1)}{\rho(x_2) - \rho(x_1)} = \frac{\Delta j}{\Delta \rho}\ . \tag{11}$$

Note that formally the fluid-dynamical description has broken down here because both ρ and j are no longer continuous functions of x.

An illustrative example is a queue, such as at a red light. When the light turns green, the outflow front quickly smoothes out, whereas the inflow front remains steep.

Note that usually at maximum flow $c = j' = 0$. Structures which operate at maximum flow do not move in space.

Leibig [70] gives results of how a random initial distribution of density steps in a closed system evolves towards two single steps according to the Lighthill-Whitham-theory.

D. Lighthill-Whitham with dissipation

Adding dissipation to the Lighthill-Whitham-equation leads to

$$\partial_t \rho + j'(\rho)\, \partial_x \rho = D\, \partial_x^2 \rho\ . \tag{12}$$

The solution of this equation is again a non-dispersive wave with phase and group velocity j'. The difference is that D introduces dissipation (damping) of the wave: The amplitude decays as e^{-Dk^2}, where k is the wavenumber. This reflects the intuitively reasonable effect that traffic jams should tend to dissolve under homogeneous and stationary conditions.

E. The nonlinear diffusion (Burgers) equation

For a further development, $j(\rho)$ has to be specified. Since we are mostly interested in the behavior of traffic near maximum throughput, we start by choosing the simplest mathematical form which yields a "well-behaved" maximum:

$$j(\rho) = v_{max}\, \rho(1-\rho) \ , \tag{13}$$

which, in traffic science, is called the Greenshields-model (see [10]). v_{max} is, in principle, a free parameter, but it has an interpretation as the maximum average velocity for $\rho \to 0$.Mathematicians would set $v_{max} = 1$; traffic scientists use $1 - \rho/\rho_{\mathrm{jam}}$ for the term in parenthesis. ρ_{jam} is the density of vehicles in a jam. The maximum current j_{max} is reached at $\rho(j_{max}) = 1/2$.

Substituting (13) into (12) yields

$$\partial_t \rho + v_{max}\, \partial_x \rho - 2 v_{max}\, \rho\, \partial_x \rho = D\, \partial_x^2 \rho \ . \tag{14}$$

Musha and coworkers [23] have shown that by introducing a linear transformation of variables

$$x = v_{max}\, t' - x' \ , \qquad t = t' \ , \tag{15}$$

one obtains

$$\partial_{t'} \rho + 2\, v_{max}\, \rho\, \partial_{x'} \rho = D\, \partial_{x'}^2 \rho \ , \tag{16}$$

which is the (deterministic) Burgers equation [71].

The transformation (15) does two things:

(1) Transformation to a coordinate system which is moving with uniform velocity v_{max}, that is, vehicles with v_{max} do not move at all in this new coordinate system, and slower vehicles move backwards (= to the left).

(2) A reversal of direction, i.e., the vehicles which are moving backwards after part (1) of the transformation now move to the right. Note that this causes a change of sign before the nonlinear term – which does not have any explanatory value except that it brings Eq. 16 *exactly* to the form treated by Burgers.

This equation has been investigated in great detail by Burgers [71] as the simplest non-linear diffusion equation. The stationary solution is a uniform density $\rho(x,t) = const.$ A single disturbance from this state evolves over time into a characteristic triangular structure with amplitude $\sim t^{-1/2}$, width $\sim t^{1/2}$, bent to the right such that the right side of the disturbance becomes discontinuous, and moving to the right with velocity $c = j' = 2\, \rho\, v_{max}$.

When interpreting this for traffic jams, one has to re-transform the coordinates. Jams can then move *both* to the left or to the right (with velocities between v_{max} and $-v_{max}$), and the discontinuous front develops at the inflow side of the jam, i.e. where the vehicles enter the jam. One sees that this solution is just the solution of the characteristics, with a dissipating diffusion term added—as should be expected because of $D > 0$.

Some other versions of the Burgers equation have been investigated thoroughly [72–74]. Of interest in the context of this paper are:

Noisy Burgers equation: Adding a Gaussian noise term η to the equation (i.e. $\langle\eta(x,t)\eta(x',t')\rangle = \eta_0\,\delta(x-x')\,\delta(t-t')$) leads to the noisy Burgers equation

$$\partial_t\rho + 2\,v_{max}\,\rho\partial_x\rho = D\partial_x^2\rho + \eta\ . \tag{17}$$

This equation does no longer converge towards a homogeneous state.

Generalized Burgers equation: The nonlinearity of the Burgers equation can be generalized:

$$\partial_t\rho = \sum_\beta b_\beta\partial_x\rho^\beta + D\partial_x^2\rho\ . \tag{18}$$

Generalized Burgers equations with arbitrary β have been investigated [73,72].

F. Including momentum

The equations so far do not explain the spontaneous phase separation into relatively free and rather dense regions of vehicles which is observed in real traffic. To obtain this, one has to include the effect of momentum: One can neither accelerate instantaneously to a desired speed nor slow down without delay. It becomes necessary to include the momentum equation. Here, one has to specify the force term F/m, which describes acceleration and slowing down. At least two properties are usually incorporated, which are called the "relaxation term" and the "interaction term".

A first order approximation for the relaxation term is [19,18]

$$\frac{1}{\tau}(V(\rho)-v), \tag{19}$$

where $V(\rho)$ is the desired average speed as a function of density, and τ is a relaxation time. This choice yields exponential relaxation towards the desired speed. The function $V(\rho)$ has to be specified externally, for example from measurements.

A commonly used interaction term [75–77,19,18] is

$$-\frac{c_0^2}{\rho}\partial_x\rho\ . \tag{20}$$

The meaning is that one tends to reduce speed when the density increases, even when the local density is still consistent with the current speed.

A more formal possible derivation of the interaction term is as follows: [2] In real traffic, the relaxation term actually is asymmetric with respect to the vehicle position, e.g., say, $\dot v(x) = \frac{1}{\tau}[V(\Delta x) - v]$ (see car-following section), where Δx is the front-buffer-to-front-buffer distance to the next vehicle ahead.

After approximating $V(\Delta x)$ by $V(\rho(x+\Delta x/2))$ and then Taylor-expanding, one obtains

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[2] I got the idea for this argument from B.S. Kerner.

$$\frac{1}{\tau}[V(\Delta x) - v] \approx \frac{1}{\tau}[V(\rho(x)) - v] + \frac{1}{\tau}\,\Delta x\, V'(\rho(x))\,\partial_x \rho \ , \tag{21}$$

thus obtaining a formal justification of the interaction term $\sim \partial_x \rho$ out of the asymmetry of the relaxation term.

c_0 is treated as constant; in traffic, a typical value for c_0 is 15 km/h [20].

Formally, the interaction term is similar to the pressure term of compressible flow, $-(1/\rho)\partial_x p$, where p is the pressure. Assuming an ideal gas ($p = \rho RT$) and isothermic behavior $T = const$, one obtains waves similar to sound waves as a solution of the linearized equations.[3] This leads to Eq. 20, where c_0 is the speed of the "sound" waves. (See below for a short discussion.)

Note that sound waves move in *both* directions from a disturbance, which means that sound waves alone are not a good explanation for freeway start-stop-waves, contrary to what is sometimes written [78].

Taking all this together, a possible momentum equation for traffic therefore is [19]

$$\partial_t v + v \partial_x v = -\frac{c_0^2}{\rho}\partial_x \rho + \frac{1}{\tau}\,[V(\rho) - v] + \nu \partial_x^2 v \ . \tag{22}$$

Since one now has two variables, one also needs an equation of continuity to close the system:

$$\partial_t \rho + \partial_x(\rho\, v) = D\, \partial_x^2 \rho \ . \tag{23}$$

Usually, D is set to zero.

For this equation, the homogeneous solution $(v, \rho) \equiv (v_0, \rho_0)$ is unstable for densities near maximum flow for a suitable choice of parameters. Using the methods of nonlinear dynamics, Kühne and coworkers [19,22,21] went beyond linear stability analysis (see also [93,80]). One finds a multitude of stable or unstable fixpoints and limit cycles which suggest that traffic near maximum flow operates on a strange attractor. This can lead to quasi-periodic behavior, exactly as is observed in traffic measurements.

Earlier work [75,18] has analyzed the same equation without viscosity ($\nu = 0$).

G. Discussion of fluid-dynamical approaches

Fluid-dynamical models have been used in traffic science for a long time, with considerable success. But they have shortcomings. Some of the major points are:

(i) One has to give externally the relation between speed or current and density. This is unsatisfying in terms of the development of a theory. But an even more intricate problem is that there is no agreement on a functional form of the speed-density relation; it is even under discussion if this relation is at all continuous [13,81].

(ii) Microscopically, temperature parameterizes the random fluctuations of particles around their mean speed: $T \propto \langle v^2 \rangle - \langle v \rangle^2$. For gases, fluctuations and therefore temperature increase with density. For granular media such as vehicular traffic or sand, fluctuations *de*crease with density (i.e. inside a jam) – it has been claimed that exactly this

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[3] True sound waves, though, would assume the gas to behave adiabatic, i.e. $p \propto \rho^\kappa$.

inverse temperature effect is responsible for clustering [82]. In this way, assuming isothermic instead of adiabatic behavior as done for the momentum equation seems only half the way one has to go. Helbing [83] discusses this further.

(iii) Helbing [83] also discusses the effect of excluded volume to take into account the spatial extension of vehicles.

(iv) Daganzo [84] claims that *all second-order* fluid-dynamical models produce unrealistic behavior (such as backwards moving vehicles caused by a diffusion term) and are therefore unsuitable for traffic science.

Nonetheless, fluid-dynamical approaches [76,19,22,21] give, for the first time, systematic insight into traffic dynamics near maximum flow beyond simple extrapolation of light and dense traffic results. These results will be further discussed near the end of this paper.

IV. DEFINITIONS OF PARTICLE HOPPING MODELS

This section defines several particle hopping models which are candidate models for traffic. They all are commonly defined on a lattice of, say, length L, where L is the number of sites. Each site can be either empty, or occupied by exactly one particle. Also, in all models particles can only move in one direction. The number of particles, N, is conserved except at the boundaries. For traffic, particles model cars.

A. The Stochastic Traffic Cellular Automaton (STCA)

The Stochastic Traffic Cellular Automaton (STCA), which has been treated in a series of papers [43–56], is defined as follows. Each particle (= car) can have an integer velocity between 0 and v_{max}. The complete configuration at time-step t is stored, and the configuration at time-step $t+1$ is computed from that, i.e. using a parallel or synchronous update. All cars/particles execute in parallel the following steps:

- Let gap = number of empty sites ahead.
- If $v > gap$ (**too fast**), then slow down to $v := gap$, [rule 1]

  else if ($v < gap$) (**enough headway**) *and* $v < v_{max}$, then accelerate by one: $v := v + 1$. [rule 2]
- **Randomization:** If after the above steps the velocity is larger than zero ($v > 0$), then, with probability p, reduce v by one. [rule 3]
- **Particle propagation:** Each particle moves v sites ahead. [rule 4]

The randomization incorporates three different properties of human driving into one computational operation: Fluctuations at maximum speed, over-reactions at braking, and retarded (noisy) acceleration.

Note that, because of integer arithmetic, conditions like $v > gap$ and $v \geq gap + 1$ are equivalent.

When the maximum velocity of this model is set to one ($v_{max} = 1$), then the model becomes much simpler: Each particle executes the following in parallel:

- If site ahead is free, move, *with probability* $1 - p$, to that site.

Since the STCA shows different behavior for $v_{max} \geq 2$ than for $v_{max} = 1$, we will distinguish them as STCA/1 and STCA/2, respectively.

Due to the given discretization of space and time, proper units are often omitted in the context of particle hopping or cellular automata models. Proper units here would be: $[gap]$ = number of cells, $[v]$ = number of cells per time step, $[t]$ = number of time steps, etc. For that reason, it is possible to write something like $v < gap$, which properly would have to be $v < gap/(\text{time step})$. Note that one still needs conversion factors to convert, say, velocity from the particle hopping model to a real world velocity, e.g. given in kilometers per hour. One should note, though, that every computer program does such a thing. Numbers in computer programs are always unitless, and a proper conversion to real world numbers has to be put in by the program designer.

B. The cruise control limit of the STCA (STCA-CC)

In the so-called cruise control limit of the STCA [51], fluctuations at free driving, i.e. at maximum speed and undisturbed by other cars, are set to zero. Algorithmically, the velocity update (rules 1 to 3) of the STCA are replaced by the following: For all cars do in parallel:

- A vehicle is stationary when it travels at maximum velocity v_{max} and has free headway: $gap \geq v_{max}$. Such a vehicle just maintains its velocity.
- Else (i.e. if a vehicle is not stationary) the standard rules 1 to 3 of the STCA are applied.

Both acceleration and braking still have a stochastic component.

C. The deterministic limit of the STCA (CA-184)

One can take the deterministic limit of the STCA by setting the randomization probability p equal to zero, which just amounts to skipping the randomization step. It turns out that, when using a maximum velocity $v_{max} = 1$, this is equivalent [72] to the cellular automaton rule 184 in Wolfram's notation [85], which is why I will use the notation CA-184/1 and CA-184/2.

Much work using CA models for traffic is based on this model. Biham and coworkers [34] have introduced it for traffic flow, with $v_{max} = 1$. Other authors base further results on it [28,36,37,39,40,43]. Some [28,40] also use it with v_{max} larger than one. It is also the basis of the two-dimensional CA models for traffic (e.g. [64–66]).

D. The cruise control version for the CA-184 (CA-184-CC)

Takayasu and Takayasu [42] introduced a different CA model which is effectively equivalent to a deterministic cruise control situation for CA-184/1. This may not be obvious from the rules, but it will become clear from the dynamic behavior summarized later. Since they use only maximum velocity $v_{max} = 1$, the rules are short: For all particles do in parallel:

- If $v = 1$ and the site ahead is free ($gap \geq 1$), then move one site ahead.

- A particle at rest ($v = 0$) can only move when $gap \geq 2$.

Generalizations to maximum velocity larger than one are straightforward, but do not seem to lead to additional insight.

E. The Asymmetric Stochastic Exclusion Process (ASEP)

The probably most-investigated particle hopping model is the Asymmetric Stochastic Exclusion Process (ASEP). Its behavior is defined as follows:

- Pick one particle randomly. [rule 1]
- If the site to the right is free, move the particle to that site. [rule 2]

The ASEP is closely related to CA-184/1 and STCA/1 (i.e. both with maximum velocity one). The difference actually only is in the manner in which sites are updated. CA-184 and STCA update all sites synchronously, whereas ASEP uses a random serial sequence.

In order to compare the ASEP with the other, synchronously updated models, one has to note that, in the ASEP, *on average* each particle is updated once after N single-particle updates. A time-step (also called update-step or iteration) in the ASEP is therefore completed after N single-particle updates ($= N$ attempted hops).

It has been noted in Ref. [72] that changing the update from asynchronous to synchronous, i.e. going from ASEP to CA-184/1, changes the dynamics considerably. In this paper, I will in addition show that re-introducing the randomness via the randomization (rule 4) in the STCA again leads to different results.

A systematic way of reducing the noise for the ASEP could be done using techniques described by Wolf and Kertesz [86], i.e. by putting a counter on each particle and move it only after k trials. For large k it becomes more and more improbably that one particle is moved twice while a neighboring particle is not moved at all during that time. Taking the limit $k \to \infty$ then reduces the ASEP to the CA-184 process in a smooth way.

One can also define higher velocities for the ASEP by simply replacing ASEP–rule 2 by STCA/2–rules 1, 2, and 4. In such a case, each particle has to remember its velocity v from the last move.

V. PARTICLE HOPPING MODELS, FLUID DYNAMICS, AND CRITICAL EXPONENTS

Both for the ASEP/1 and for the CA-184/1, fluid-dynamical limits and critical exponents are well known (see, e.g., [74,72,87,73]). The most straightforward way to put the concept of critical exponents into the context of traffic flow is to consider "disturbances" (i.e. jams) of length x and ask for the time t to dissolve them. For example, one would intuitively assume that a queue of length x at a traffic light which just turned green would need a time t proportional to x until everybody is in full motion. By this argument, the dynamic exponent z, defined by $t \sim x^z$, should be one.

Yet, there may be more complicated cases. Imagine again a queue at a traffic light just turned green but this time there is also some fairly high inflow at the end of the queue. The jam-queue itself will start moving backwards, clearing its initial position in time $t \sim x$. However, the dissolving of the jam itself may be governed by different rules. An example for this will be given in the following.