

Scale-invariant correlated probabilistic model yields q -Gaussians in the thermodynamic limit

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Extremization of the Boltzmann-Gibbs (BG) entropy $S_{BG} = -k \int dx p(x) \ln p(x)$ under appropriate norm and width constraints yields the Gaussian distribution $p_G(x) \propto e^{-\beta x^2}$. Also, the basic solutions of the standard Fokker-Planck (FP) equation (related to the Langevin equation with *additive noise*), as well as the Central Limit Theorem attractors, are Gaussians. The simplest stochastic model with such features is $N \rightarrow \infty$ *independent* binary random variables, as first proved by de Moivre and Laplace. These well known results mathematically ground BG statistical mechanics. What happens for strongly correlated random variables? Such correlations are often present in physical situations, which are often characterized by q -Gaussians, $p_q(x) \propto [1 - (1 - q)\beta x^2]^{1/(1-q)}$ [$p_1(x) = p_G(x)$]. It is typically so if we allow the Langevin equation to include *multiplicative noise*, or the FP equation to be *nonlinear*. The ubiquitous property of scale-invariance enables a systematical analysis of the relation between correlations and non-Gaussian distributions. Nevertheless, a generalized stochastic model yielding q -Gaussians ($q \neq 1$) was missing. This is achieved here by using the Laplace-de Finetti representation, which embodies strict scale-invariance of interchangeable N random variables. We also demonstrate that strict scale invariance together with q -Gaussianity mandates the associated *extensive* entropy to be that of BG.

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One of the cornerstones of statistical mechanics is the functional connection of the thermodynamic entropy with the set of probabilities $\{p_i\}$ of microscopic configurations. For the celebrated Boltzmann-Gibbs (BG) theory, this central functional is given by $S_{BG} = -k \sum_{i=1}^W p_i \ln p_i$, where W is the total number of microscopic states which are compatible with the information that we have about the system. This powerful connection is in principle applicable to a vast class of relevant systems, including (classical) dynamical ones whose maximal Lyapunov exponent is positive, thus generically warranting strong chaos, hence mixing in phase space, hence ergodicity (Boltzmann, in some sense, embodied all these features in his insightful *molecular chaos hypothesis*). Within this theory, it ubiquitously emerges the Gaussian distribution $p_G \propto e^{-\beta x^2}$ ($\beta > 0$). Indeed, this important probabilistic form (i) maximizes the (continuous version of the) entropy $S_{BG} = -\int dx p(x) \ln[p(x)]$ under the basic constraints of normalizability and finite width; (ii) constitutes the exact solution, for all values of space and time, of the simplest form of the (linear) Fokker-Planck equation, in turn based on the simplest form of the Langevin equation (which includes *additive noise*); (iii) is the $N \rightarrow \infty$ attractor of the (appropriately centered and scaled) sum of N *independent* (or weakly correlated in an appropriate sense) discrete or continuous random variables whose second moment is finite (Central Limit Theorem); (iv) is the velocity distribution (Maxwell distribution) of any classical many-body Hamiltonian system whose canonical (thermal equilibrium with a thermostat) partition function is finite i.e.,

if the interactions between its elements are sufficiently short-ranged, or inexistent. The simplest probabilistic model which realizes these paradigmatic features is a set of N independent equal binary random variables (each of them taking say the values 0 and 1 with probability 1/2). The probability of having, for fixed N , n 1's is given by $\frac{N!}{n!(N-n)!} 2^{-N}$. Its limiting distribution is, after centering and scaling, a Gaussian (as first proved by de Moivre and Laplace), and its (extensive) entropy is the one of BG, since $S_{BG}(N) = Nk \ln 2$.

What happens with the above properties when the correlations between say the elements of a probabilistic model are strong enough (in the sense that they spread over all elements of the system)? There is in principle no reason for expecting the relevant limiting distribution to be a Gaussian, and the entropy which is extensive (i.e., $S(N) \propto N$ for $N \gg 1$) to be S_{BG} . The purpose of the present paper is to answer such questions for a class of systems which are ubiquitous in natural, artificial and even social systems, namely those which are *scale-invariant* in a probabilistic sense which we shall define below. Let us now discuss the ubiquitous appearance of q -Gaussians, defined as

$$p_q(x) \propto [1 - (1 - q)\beta x^2]^{1/(1-q)} \quad [p_1(x) = p_G(x)] \quad , \quad (1)$$

$x \in R$ for $q < 3$ (for $q \geq 3$, normalizability is lost), and $x^2 \leq 1/[(1 - q)\beta]$ for $q < 1$. q -Gaussians, which display asymptotic power-laws, can be seen as the natural generalization of the Gaussian. (i) q -Gaussians appear as the exact solutions of paradigmatic non-Markovian Langevin processes and their associated Fokker-Planck

equations. Langevin equations with both additive and multiplicative noise [1] or Langevin equations with long-range-memory [2], lead to inhomogeneous linear [3], or homogeneous nonlinear [4, 5] Fokker-Planck equations, respectively. (ii) q -CLT attractors are q -Gaussians [6]. (iii) The extremization of S_q with norm and finite width constraints yields q -Gaussians. S_q is a generalization of BG entropy, namely [7]

$$S_q = k \frac{1 - \int dx [p(x)]^q}{q - 1} \quad (q \in R; S_1 = S_{BG}) \quad (2)$$

This entropy is generically *nonadditive* [8]. However, for many systems a value of q , denoted by q_{ent} , exists for which $S_{q_{\text{ent}}}$ is *extensive*, i.e., $S_{q_{\text{ent}}}(N) \propto N$ ($N \gg 1$). As is well known, for all standard short-range-interacting many-body Hamiltonian systems, we have $q_{\text{ent}} = 1$. However, some systems exist for which $q_{\text{ent}} < 1$, [9, 10]. (iv) Numerical indications [11] for the distributions of velocities in quasistationary states of long-range Hamiltonians [12] suggest q -Gaussians. Further, experimental and observational evidence for q -Gaussians exists for the motion of biological cells [13, 14], defect turbulence [15], solar wind [16, 17], cold atoms in dissipative optical lattices [18], dusty plasma [19], among others. Numerical indications are also available at the edge of chaos of unimodal maps [20]. The consistently generalized simple probabilistic model was still missing for arbitrary values of q . This is what we present in this paper.

In the following we consider a binary exchangeable stochastic process, with correlated elements say from $x \in \{0, 1\}$. Exchangeable means that the N -point probabilities $p_N(x_1, x_2, \dots, x_N)$ of the process are totally symmetric in its arguments. The probability of a *specific* sequence (x_1, x_2, \dots, x_N) , a microstate, where n events are in one state, e.g. $x_i = 0$, and $N - n$ events in the other, $x_j = 1$, we denote as r_n^N , following the notation in [21]. There are $\binom{N}{n}$ such micro-states (where the order of the process is exchanged), and the probability of finding *any* situation with n events in one state is given by

$$p_n^N = \binom{N}{n} r_n^N, \quad \sum_{n=0}^N p_n^N = 1. \quad (3)$$

Total symmetry of the micro-states, $\forall N$, implies $r_n^{N-1} = r_n^N + r_{n+1}^N$, sometimes referred to as the *Leibnitz triangle rule* [9], or *scale-invariance* of the distribution. Scale-invariance means that the marginal joint distribution function where one variable is integrated out (at 'scale' $N - 1$), $p_{N-1}(x_1, x_2, \dots, x_{N-1}) = \int dx_N p_N(x_1, x_2, \dots, x_N)$ is of the same exchangeable functional form as $p_N(x_1, x_2, \dots, x_N)$.

For dealing with correlations among binary variables from totally symmetric distributions, the following representation was suggested by Laplace in 1774, and later rediscovered by de Finetti [22, 23].

$$r_n^N = \int_0^1 dy y^n (1-y)^{N-n} g(y) \quad \left[\int_0^1 dy g(y) = 1 \right]. \quad (4)$$

This representation ensures Leibnitz triangle rule and normalization, $\sum_{n=0}^N \binom{N}{n} r_n^N = 1$. Note that the non-negative g introduces the correlations in the stochastic process. In the case of independent variables, we have $g(y) = \delta(y - p)$, where p is the probability for one of the states, say $x = 0$. The fact that *any* exchangeable stochastic process can be represented in this way is called the Laplace-de Finetti theorem [22–24].

Defining $\rho_n^N \equiv (N+1)p_n^N$ and using properties of the Beta function, $B(a, b) \equiv \int_0^1 dx x^{a-1} (1-x)^{b-1}$, one obtains

$$\rho_n^N = \frac{\int_0^1 dx x^n (1-x)^{N-n} g(x)}{\int_0^1 dx x^n (1-x)^{N-n}}. \quad (5)$$

The normalization condition thus becomes $1 = \sum_{n=0}^N (N+1)^{-1} \rho_n^N$. Interpreting this sum in a Riemann-Stieltjes sense the limit distribution $\rho(y)$ on $y \in [0, 1]$ can be constructed by identifying $dy = 1/(N+1)$. We use the discretization $y_n^N = (1/2 + n)/(N+1)$ for $n = 0, 1, \dots, N$ of the unit-interval $[0, 1]$. Denote the natural number closest to the value $(N+1)y - 1/2$ by $[(N+1)y]$ then the limit distribution gets $\rho(y) = \lim_{N \rightarrow \infty} \rho_{[(N+1)y]}^N$. On the other hand this means that $\rho(y) = \lim_{N \rightarrow \infty} \frac{\int_0^1 dx [x^y (1-x)^{1-y}]^N g(x)}{\int_0^1 dx [x^y (1-x)^{1-y}]^N}$. Notice that the maximum of $x^y (1-x)^{1-y}$ with respect to x is obtained at $x = y$, and for $N \rightarrow \infty$ we get $\delta(x - y) = \lim_{N \rightarrow \infty} \frac{[x^y (1-x)^{1-y}]^N}{\int_0^1 dx [x^y (1-x)^{1-y}]^N}$. Consequently, the limit distribution ρ can be precisely identified with the function g ,

$$\rho(y) = \int_0^1 dx \delta(x - y) g(x) = g(y), \quad (6)$$

i.e. $\lim_{N \rightarrow \infty} \sum_n |\rho_n^N - g(y_n^N)| = 0$. The limit distribution and the function generating the Leibnitz-triangle r_n^N are in fact identical! Therefore, once a desired limiting distribution is given, one can simply write down the sequence of r_n^N which is generating it. This interpretation of g makes clear why non-negativity of g is required. The limit distribution $\rho(y)$, i.e. $g(y)$, of binary exchangeable processes is defined on $y \in [0, 1]$, where y is the ratio of events n/N in the limit of large N . In particular the discrete $y_n^N \in [0, 1]$ and are different from 1 and 0. However, prototypical processes, e.g. spin-systems or random walks, depend on binary processes (spins up/down or number of left/right steps). Yet, their associated observable variable is not the ratio n/N of binary events but some other descriptive variable z (the magnetization or the position of the random walker). The distribution

$G(z)$, which we assume to be symmetric, of the descriptive variable z can be obtained by a transformation of variables defined by $dzG(z) = dy\rho(y)$. Such symmetric distributions $G(z)$ have domains different from $[0, 1]$ (e.g. typically $[-1, 1]$ for spin systems and $[-\infty, \infty]$ for random walks). We will call a strictly monotonous increasing antisymmetric functions f , such that the effective stochastic variable z will take the values $z_n^N = f(2y_n^N - 1)$, a symmetric representation of the binary process. To be clear, $z = f(2y - 1)$ is precisely the change of variables relating the generating function g of the binary process with the limit distribution G of the observable process. In particular, G and g are one-to-one related by $2f'(2y - 1)G(f(2y - 1)) = g(y)$. Each pair (f, g) exactly defines one representation G of the binary process. Moreover, fixing G and the representation f uniquely determines g . Inversely, for a given observable distribution G , any pair (f, g) that represents G can serve as a stochastic model for G .

We now proceed to derive a stochastic model with q -Gaussian limit distributions. For the case of $q < 1$, the q -Gaussian $G_q(z) = \frac{1}{Z_q}[1 - (1 - q)z^2]^{1/(1-q)}$ is defined on a compact support, $|z| \leq \frac{1}{\sqrt{1-q}}$. To identify it with the limiting distribution we have to map z to the unit interval by $z = f(2y - 1) = \frac{1}{\sqrt{1-q}}(2y - 1)$. Since the support of the q -Gaussian $G_q(z)$ for $q < 1$ is compact an affine transformation is the natural choice for the change of variables. Under an affine variable transformation $G_q(z) \rightarrow \rho(y) = \frac{1}{Z_q}4y^{\frac{1}{1-q}}(1 - y)^{\frac{1}{1-q}}$, where $Z_q = 4B\left(\frac{1}{1-q} + 1, \frac{1}{1-q} + 1\right)$. Consequently, by introducing the notation $\nu \equiv \frac{1}{1-q} + 1$, we get $g(y) = \frac{[y(1-y)]^{\frac{1}{1-q}}}{B(\nu, \nu)}$ and, finally, by using Eq. (4),

$$r_n^N(q) = \frac{B(\nu + n, \nu + N - n)}{B(\nu, \nu)} \quad (7)$$

To retrieve the original q -Gaussian $G_q(z)$ one has to perform the inverse coordinate transformation. This maps the discretization $z_n^N = f(2y_n^N - 1)$, i.e. $z_n^N = (2y_n^N - 1)/\sqrt{1-q}$ and the discretization-width becomes $dz \equiv 2/(\sqrt{1-q}(N+1))$, which takes into account the factor $2f'(2y_n^N - 1) = 2/\sqrt{1-q}$ from the change of variables. Analogous to ρ the discrete formulation of the q -Gaussian G_q reads $F_n^N = (dz)^{-1} \binom{N}{n} r_n^N$ and $\lim_{N \rightarrow \infty} \sum_n |F_n^N - G_q(z_n^N)| \rightarrow 0$. This is exactly the model which was heuristically found in [21]. Note that there the model was presented as $r_0^N(q) = \frac{(N+\nu-1)(N+\nu-2)\dots\nu}{(N+2\nu-1)(N+2\nu-2)\dots(2\nu)}$, for integer values of ν , which is obviously a particular case of Eq. (7).

Since the q -Gaussian has no compact support for $q > 1$ the situation becomes more involved, since now one has to map the real axis to the unit interval $[0, 1]$. Such a q -Gaussian $G_q(z)$ with $z \in [-\infty, \infty]$ may be thought of as distributions of the distance z a peculiar random walk

has covered in the long time limit. There is no map f now that is a natural choice, a priori. However, in order to explicitly compute the probabilities r_n^N one may choose a map such that r_n^N is given in terms of Beta functions, as before. This leads to the situation that pairs of q -Gaussians, one with $q < 1$ and another with $q > 1$, can be generated by the same binary process and differ only in terms of the representation f .

Let us find a map f such that $f(2y - 1)$ again maps $y \in [0, 1]$ to $z \in [-\infty, \infty]$. Using the normalization condition of the q -Gaussian this variable transformation implies $1 = \int_{-\infty}^{\infty} dz G_q(z) = \int_{-1}^1 dx f'(x) G_q(f(x))$ and identify

$$g(y) = 2f'(2y - 1)G_q(f(2y - 1)) \quad (8)$$

A particular f to compute r_n^N in closed form is given by

$$f(y) = \frac{y}{\sqrt{1-y^2}} \frac{1}{\sqrt{q-1}} \quad \text{and} \quad f'(y) = \frac{(1-y^2)^{-3/2}}{\sqrt{q-1}} \quad (9)$$

Noting, that this model f for the q -Gaussian implies $[1 - (1-q)f(y)^2]^{1/(1-q)} = (1-y^2)^{1/(q-1)}$. Inserting this into Eq. (8) we finally get the stochastic model,

$$r_n^N(q) = \frac{B(\mu + n, \mu + N - n)}{B(\mu, \mu)} \quad (10)$$

where we have used the notation $\mu = \frac{1}{q-1} - \frac{1}{2}$. All that is left is to place everything correctly. We use the same discretization $y_n^N = (1/2 + n)/(N+1)$ for the interval $[0, 1]$ as for $q < 1$. Again, $dy_n^N = 1/(N+1)$ is the width of the discretization on $[0, 1]$ and with $\rho_n^N = (dy_n^N)^{-1} \binom{N}{n} r_n^N$, we get that $\lim_{N \rightarrow \infty} \sum_n |\rho_n^N - g(y_n^N)| = 0$, as a power of N as in the $0 < q < 1$ case. To retrieve the q -Gaussian $G_q(z)$, one has to perform the inverse coordinate transformation $y \in [0, 1] \rightarrow z \in [-\infty, \infty]$. This is a little more complicated since the discretization width now depends on N and n . In particular, mapping back to $[-\infty, \infty]$ gives the discretization $z_n^N = f(2y_n^N - 1)$, i.e. $z_n^N = (y_n^N - 1/2)/\sqrt{(q-1)y_n^N(1-y_n^N)}$, and the discretization width becomes $dz_n^N \equiv (y_n^N(1-y_n^N))^{-3/2}/(4(N+1)\sqrt{q-1})$. The discretized version of the q -Gaussian for $q > 1$ now reads, $F_n^N = (dz_n^N)^{-1} \binom{N}{n} r_n^N$. In the limit we get $\lim_{N \rightarrow \infty} \sum_n |F_n^N - G_q(z_n^N)| \rightarrow 0$, as a power of N . Comparing Eq. (7) and Eq. (10) it is obvious that for the two models of the q -Gaussian distribution ($q < 1$ and $\bar{q} > 1$) the generating binary processes r_n^N are identical whenever $\nu = \mu$. ν and μ are functions of q and \bar{q} , and $\nu(q) = \mu(\bar{q})$ establishes a relation between two q -Gaussian distributions generated by an identical exchangeable binary stochastic process, i.e. $\bar{q} = \frac{7-3q}{5-3q}$; q increasing from $-\infty$ to 1 yields \bar{q} decreasing from $5/3$ to 1. Therefore, the models of q -Gaussian distributions with $q < 1$ are conjugate with the model of \bar{q} -Gaussian with $\bar{q} \in [1, 5/3]$ in the sense of being driven by an identical binary stochastic process. The class of

\bar{q} -Gaussian distributions with $\bar{q} \in (5/3, 3]$, which is exactly the class of normalizable \bar{q} -Gaussian distributions with diverging second moment, are not identified with a $q < 1$. The corresponding binary processes are unique in this sense. For instance, choosing $1 < \bar{q} = 2$ requires $1 > q = 3$, which is impossible. For $\bar{q} = 2$ it follows that $\mu = 1/2$ and the associated binary process $r_n^N = B(0.5 + n, 0.5 + N - n) / B(0.5, 0.5)$ has no representation for any $q < 1$, and $r_0^N = \frac{B(0.5, 0.5 + N)}{B(0.5, 0.5)} = 2^{-2N} \binom{2N}{N}$. Of course one can choose many families (f_q, g_q) of models for q -Gaussian limit distributions and, for each strictly monotonous function $\bar{q} : [-\infty, 1] \mapsto [1, 3]$, it is possible to construct *conjugate* families of models in the sense that they are generated by the same family of binary processes. In fact when the function L_q is defined by $\int_0^{L_q(z)} dz' G_q(z') = \int_0^z dz'' G_{\bar{q}(q)}(z'')$ and (f_q, g_q) is a model for a q -Gaussian with $q < 0$ then $(L_q(f_q), g_q)$ is a conjugate model of the \bar{q} -Gaussian with $\bar{q} = \bar{q}(q)$. This allows to conjugate families of models for different dualities recognized in q -statistics, e.g. $\bar{q} = (5 - 3q)/(3 - q)$ for q -Fourier transforms [25].

We have explicitly derived two possible stochastic models for correlated and exchangeable binary random variables, which lead to exact q -Gaussians as the limiting distributions, while *strictly* satisfying Leibnitz rule,

$$r_n^N = \begin{cases} \frac{B\left(\frac{2-q}{1-q} + n, \frac{2-q}{1-q} + N - n\right)}{B\left(\frac{2-q}{1-q}, \frac{2-q}{1-q}\right)} & \text{if } q < 1, \\ 1/2^N & \text{if } q = 1, \\ \frac{B\left(\frac{3-q}{2q-2} + n, \frac{3-q}{2q-2} + N - n\right)}{B\left(\frac{3-q}{2q-2}, \frac{3-q}{2q-2}\right)} & \text{if } 1 < q < 3, \end{cases} \quad (11)$$

$q = 1$ can be obtained by both $q \rightarrow 1^\pm$. Eq. (11) is valid for the affine representation for $q < 1$ and for the representation f given in Eq. (9) for $q > 1$. For different families of representations f_q of the q -Gaussians the equation has to be adapted. In the $q < 1$ case the model for the q -Gaussian is unique in the sense that the representation f of the binary model is the affine map from $[0, 1]$ to the domain of the q -Gaussian. Models which only *asymptotically* satisfy the Leibnitz rule are infinite in number: see [21] for two $q > 1$ examples.

With the Laplace-de Finetti representation we consider the entropy S_q for large N and prove that $q_{\text{ent}} = 1$:

$$S_Q[g] = \frac{1 - \sum_{n=0}^N \binom{N}{n} [r_n^N]^Q}{Q - 1} = \frac{\int_0^1 dy \left[\int_0^1 dx [x^y (1-x)^{1-y}]^N \right]^{Q-1} [g(y)]^Q - 1}{Q - 1}. \quad (12)$$

Extensivity requires $\frac{dS_{q_{\text{ent}}}[g]}{dN}$ to be a positive constant for $N \gg 1$. We find, by using Stirling's approximation, $\frac{dS_Q[g](N)}{dN} \sim 2^{N(1-Q)} g(1/2) \ln 2$. Hence, for any

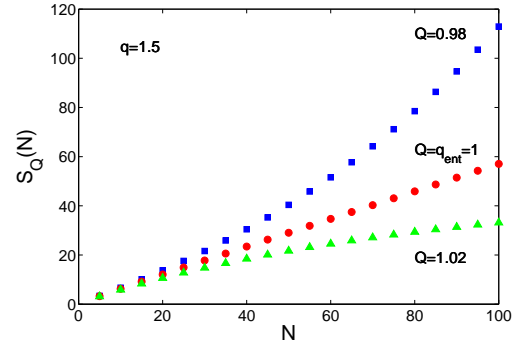


FIG. 1: Nonadditive entropy S_Q vs. N for a q -Gaussian with $q = 1.5$, for $Q = 0.92, 1$ and 1.02 . Clearly, only $Q = 1$ is extensive (i.e., $S_1(N) \propto N$, for $N \gg 1$).

$g(1/2) > 0$, it must be $Q = 1$, i.e., $q_{\text{ent}} = 1$ ($\forall q$). Since this result depends only on r_n^N (and not on f), $q_{\text{ent}} = 1$ is true for all such models.

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