

A Note on Fundamental, Non-fundamental, and Robust Cycle Bases

Konstantin Klemm^a and Peter F. Stadler^{a,b,c}

^a*Bioinformatics Group, Department of Computer Science
and Interdisciplinary Center for Bioinformatics
University of Leipzig, Härtelstraße 16-18, D-04107 Leipzig, Germany*

^b*Department of Theoretical Chemistry
University of Vienna, Währingerstraße 17, A-1090 Wien, Austria*

^c*Santa Fe Institute,
1399 Hyde Park Rd., Santa Fe, NM 87501, USA*

Abstract

In many biological systems, robustness is achieved by redundant wiring, and reflected by the presence of *cycles* in the graphs connecting the systems' components. When analyzing such graphs, *cyclically robust cycle bases* are of interest since they can be used to generate all cycles of a given 2-connected graph by iteratively adding basis cycles. It is known that *strictly fundamental* (or *Kirchhoff*) *bases*, i.e., those that can be derived from a spanning tree, are not necessarily cyclically robust. Here we note that, conversely, cyclically robust bases (even of planar graphs) are not necessarily fundamental. Furthermore, we present a class of cubic graphs for which cyclically robust bases can be explicitly constructed.

Key words: Cycle basis, fundamental cycle basis, cyclically robust basis

1 Introduction

Much insight into complex biological systems is gained by the study of the underlying networks that connect the systems' interacting substrates. Among other properties, the ubiquitous robustness of cells and organisms against noise and mutations is often reflected in the structure of these networks. For instance, in an extensive systematic study of signaling pathways, Wagner and Wright [14] recently found that alternative pathways between regulators and targets are the rule rather than the exception. These multiple paths between two vertices in the networks give rise to *cycles* in the interaction network. As larger and larger biological systems are mapped into networks, the need for efficient analysis tools, including cycle detectors, increases.

So-called “cycle-space algorithms” attempt to construct the set of all circuits of a graph from a cycle basis \mathcal{B} by iteratively computing the symmetric difference of a circuit and a basis cycle, subsequently retaining the result if and only if it is again a circuit. A cycle basis is cyclically robust if this approach is successful (see [4, 8], or the discussion in Section 4 below, for a more ‘rigorous’ definition of this concept). Cyclically robust bases are therefore of interest as a computational tool in network analysis. Unfortunately, however, very little is known about their structural properties beyond a few special graph classes: As shown in [4], the boundaries of the faces of an embedded planar graph form a cyclically robust cycle basis. Corresponding cycle-space algorithms are given in [13, 4]. Furthermore, complete graphs and complete bipartite graphs have cyclically robust bases that are easy to construct explicitly [8]. On the other hand, there is at present no efficient algorithm to construct, for just any given graph as an input graph, a corresponding cyclically robust cycle basis. Indeed, it is still unknown whether cyclically robust bases always exist. In [10], this problem is at least partially circumvented by employing larger generating sets instead of cycle bases.

As a first step towards a better understanding of the structure of cyclically robust bases, we study here their relationships with other classes of cycle bases that have been explored in much more detail in the past [6, 11]. In particular, we consider *fundamental* bases (which are related to ear decompositions, cf. [16]) and *strictly fundamental* or *Kirchhoff* bases (which can be obtained from spanning trees, cf. [9]). In [12, 4], it was shown that Kirchhoff bases and, hence, fundamental bases are not necessarily cyclically robust. Here, we first deal with the converse question: *Is a cyclically robust basis always fundamental?* and show, by systematically studying non-fundamental bases of graphs with small cyclomatic number, that the answer to this question is, in general, also negative.

In addition, we treat a class of non-planar cubic graphs. We show that each member of this infinite set of graphs has a fundamental and cyclically robust cycle basis. In the last section, we summarize all what appears to be known about the (lack of) mutual relationship between the various classes of cycle bases in form of a simple diagram, cf. Figure 5.

2 Cycles, Circuits, and Cycle Bases

Throughout this contribution, let $G = (V, E)$ be a finite undirected simple 2-connected graph. A (*generalized*) *cycle* in G is an Eulerian subgraph of G , i.e., a subgraph of G in which the degree of every vertex is even. A connected Eulerian subgraph in which every vertex has degree 2 will be called an *elementary cycle* or a *circuit*. In the following, we will identify a subset $E' \subseteq E$ of edges of G

with the subgraph $G(E') := (\cup E', E')$ of G that it defines. In particular, we identify cycles with their edge sets. The symmetric difference of two edge sets E' and E'' will be denoted by $E' \oplus E''$, i.e., we put $E' \oplus E'' := (D \cup D') \setminus (D \cap D')$.

The power set $\mathfrak{P}(E)$ can be regarded as a vector space over $\text{GF}(2) = \{0, 1\}$ with vector addition \oplus and the trivial multiplication operator $1 \cdot D = D$, $0 \cdot D = \emptyset$. The *cycle space* $\mathfrak{C}(G)$ is the subspace of $(\mathfrak{P}(E), \oplus, \cdot)$ that consists of the cycles of G (including the “empty cycle” \emptyset), see e.g. [2]. As every 2-connected graph G is connected, the dimension $\dim_{\text{GF}(2)} \mathfrak{C}(G)$ of its cycle space coincides with its cyclomatic number $\mu(G) := |E| - |V| + 1$ [5].

A basis \mathcal{B} of $\mathfrak{C}(G)$ that consists of circuits, only, is a *cycle basis* of G . For every cycle C , there is a unique subset $\mathcal{B}_C \subseteq \mathcal{B}$ of circuits in \mathcal{B} such that $C = \bigoplus_{C' \in \mathcal{B}_C} C'$ holds.

It is well known and easy to see that the collection of cycles formed by the (boundaries of the) faces of an embedded planar graph G is a cycle basis of G . Any such cycle basis is called a *planar* cycle basis of G . Further, a cycle basis \mathcal{B} is called *fundamental* [7, 16] if there exists a linear order “ $\preceq_{\mathcal{B}}$ ” defined on \mathcal{B} such that no circuit $C \in \mathcal{B}$ is contained in the union over all circuits $C' \in \mathcal{B}$ that are properly smaller than C relative to $\preceq_{\mathcal{B}}$, i.e.,

$$C \setminus \left(\bigcup_{C' \in \mathcal{B}, C' \prec_{\mathcal{B}} C} C' \right) \neq \emptyset$$

holds for every $C \in \mathcal{B}$.

Given (the set of edges of) a spanning tree T of G and an edge $e \in E \setminus T$, there is unique circuit $\text{cyc}(T, e)$ in $T \cup \{e\}$. The set

$$\mathcal{B}_T := \{\text{cyc}(T, e) \mid e \in E \setminus T\}$$

is a cycle basis [9], and every cycle basis of this form is called a *strictly fundamental* cycle basis. Alternatively, the strictly fundamental cycle bases can be characterized as follows [12]:

A cycle basis \mathcal{B} is strictly fundamental if and only if

$$C \setminus \left(\bigcup_{C' \in \mathcal{B} \setminus \{C\}} C' \right) \neq \emptyset \tag{1}$$

holds for all $C \in \mathcal{B}$.

Equivalently, \mathcal{B} is strictly fundamental if and only if

$$C \setminus \left(\bigcup_{C' \in \mathcal{B}, C' \prec_{\mathcal{B}} C} C' \right) \neq \emptyset$$

holds, for all $C \in \mathcal{B}$, for every linear order “ $\prec_{\mathcal{B}}$ ” of \mathcal{B} [6].

3 Graphs of Small Cyclomatic Number

3.1 Graphs of Cyclomatic Number 2

A 2-connected graph $G = (V, E)$ of cyclomatic number 2 consists of two distinct vertices connected by three paths. This follows from the fact that a 2-connected graph G with $\mu(G) = 2$ has an “open ear decomposition”, i.e., G consists of a cycle and a path attached to two disjoint vertices of that cycle [15]. The cycle space of G , thus, consists of the empty cycle and exactly 3 circuits, any two of which may be chosen to form the boundaries of the two faces of a planar embedding while the third one is formed by its *circumference*. Further, a subgraph T of G is a spanning tree if and only if (i) it contains all edges of G but two and (ii) the two missing edges are contained in two distinct of the three paths constituting G . Thus, the circuit $\text{cyc}(T, e)$ obtained from T by adding one of those two missing edges e coincides exactly with the unique circuit of G not containing the other missing edge. Thus, every basis of a 2-connected graph of cyclomatic number 2 is strictly fundamental as well as planar and, hence (cf. [4]), cyclically robust.

Note also that, given two circuits C_1 and C_2 in an arbitrary graph G , their union $C_1 \cup C_2$ is a 2-connected graph of cyclomatic number 2 if and only if their intersection $C_1 \cap C_2$ is a (non-empty) path and that, in this case, their sum $C_1 \oplus C_2$ is also a circuit while, in contrast, the assumption that $C_1 \oplus C_2$ is a circuit does not imply that $C_1 \cap C_2$ is a path (see Fig. 1 for a counter-example).

3.2 Graphs of Cyclomatic Number 3

Lemma 1 *Every cycle basis \mathcal{B} of a graph G of cyclomatic number $\mu(G) \leq 3$ is fundamental.*

Proof. As $C \setminus C' \neq \emptyset$ holds for any two distinct circuits C, C' of G , there is nothing to show in case $\mu \leq 2$.

Now, assuming that $\mu(G) = \#\mathcal{B} = 3$ holds, it suffices to show that there is an edge e that is covered by only one of the three cycles of \mathcal{B} . Otherwise, however, every edge must be covered at least twice implying that, at least, one edge $e = \{u, w\}$ is covered thrice: If every edge were covered exactly twice, the sum of all cycles were 0, contradicting linear independence. Yet, continuing the 3 circuits in \mathcal{B} running through e in one direction, say from u to w to \dots , there

must — sooner or later — come a last vertex v that is contained in all three circuits after which at least one of the three circuits diverges from the other two. Consequently, the next edge covered by this circuit is necessarily covered by only one of the three circuits in \mathcal{B} . \square

3.3 Graphs of Cyclomatic Number 4

Apparently, the argument above shows that any three linearly independent circuits of a graph G as above can be indexed as C_1, C_2, C_3 so that $C_2 - C_1 \neq \emptyset$ and $C_3 - (C_1 \cup C_2) \neq \emptyset$ holds. Moreover, extending the above argument to graphs of cyclomatic number 4 yields that there exists, up to isomorphism, a unique smallest graph $G = G = (V, E)$ of cyclomatic number 4 for which a non-fundamental cycle basis \mathcal{B} exists, viz., the **four-wheel**, that is, the graph W_4 with the 5-point vertex set $V^{(4)} := \{*, 1, 2, 3, 4\}$ and edge set

$$E^{(4)} := \left\{ \{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}, \{*, 1\}, \{*, 2\}, \{*, 3\}, \{*, 4\} \right\}.$$

Its — also unique — cycle basis $\mathcal{B}^{(4)}$ that covers every edge at least twice is the cycle basis $\mathcal{B}^{(4)} := \{C_1, C_2, C_3, C_4\}$ where C_i is defined, for $i = 1, 2, 3, 4$, by $C_i := \{\{*, i\}, \{i, i+1\}, \{i+1, i+2\}, \{i+2, i+3\}, \{i+3, *\}\}$ (with sums computed modulo 4).

4 Cyclically Robust Cycle Bases

We basically follow the notation introduced in [8]. A sequence (C_1, C_2, \dots, C_k) of circuits is defined to be *cyclically well-arranged* if each partial sum

$$Q_j = \bigoplus_{i=1}^j C_i \tag{2}$$

is a circuit for all $j \leq k$, and it is defined to be *strictly well-arranged* if

$$C_j \cap \left(\bigoplus_{i=1}^{j-1} C_i \right) = C_j \cap Q_{j-1} \tag{3}$$

is a path. Apparently, strictly well-arranged cycle sequences are also cyclically well-arranged while the converse is not true (cf. [8] or Section 3.1 and Fig 1).

A cycle basis \mathcal{B} is called *cyclically* (or, respectively, *strictly*) robust if, for every circuit C , the corresponding set \mathcal{B}_C can be cyclically (strictly) well-arranged.

Note that, associating to a cycle basis \mathcal{B} the directed graph $\Gamma_{\mathcal{B}}$ whose vertex set is the collection $\mathbf{P}_{\mathcal{B}}$ of subsets \mathcal{B}' of \mathcal{B} that are of the form \mathcal{B}_C for a cycle $C \subseteq E$ that is either the empty cycle or a circuit (or, equivalently, the collection of subsets \mathcal{B}' of \mathcal{B} for which $\bigoplus \mathcal{B}' := \bigoplus_{C' \in \mathcal{B}'} C'$ is either the empty cycle or a circuit) while its edge set $\mathbf{E}_{\mathcal{B}}$ consists of all (ordered) pairs $(\mathcal{B}', \mathcal{B}'')$ of subsets of \mathcal{B} in $\mathbf{P}_{\mathcal{B}}$ for which $\mathcal{B}'' = \mathcal{B}' \cup \{C''\}$ holds for some circuit $C'' \in \mathcal{B} - \mathcal{B}'$, the cycle basis \mathcal{B} is cyclically robust if and only if this graph is ‘connected relative to \emptyset ’, i.e., there exists a directed path from its vertex \emptyset .

So, to explicitly check whether a given cycle basis \mathcal{B} is cyclically robust, one just needs to compute all the subsets of \mathcal{B} that are of the form \mathcal{B}_C for a circuit $C \subseteq E$.

A similar criterion apparently also holds for strictly robust cycle bases: One just needs to replace $\mathbf{E}_{\mathcal{B}}$ by its subset $\mathbf{E}'_{\mathcal{B}}$ consisting of all pairs $(\mathcal{B}', \mathcal{B}'')$ in $\mathbf{E}_{\mathcal{B}}$ for which $\mathcal{B}'' = \mathcal{B}' \cup \{C''\}$ holds for some circuit $C'' \in \mathcal{B} - \mathcal{B}'$ for which, in addition, the intersection $C'' \cap \bigoplus \mathcal{B}'$ is a path.

Dixon and Goodman [3] conjectured that every strictly fundamental cycle basis is cyclically robust. A counter example, however, was given in [12]. Later, it was shown that, as mentioned already in the introduction, any planar cycle basis is cyclically [4] (and even strictly [8]) robust.

In [8], a cyclically robust basis of K_4 is given that is not strictly robust. The same holds for the unique non-fundamental cycle basis $\mathcal{B}^{(4)}$ of the four-wheel W_4 discussed above: Indeed, in addition to the four circuits $\{C_1, C_2, C_3, C_4\}$ in $\mathcal{B}^{(4)}$, there are — with i running from 1 to 4 and addition taken modulo 4 — four circuits of the form $C_{(i,i+2)} := \{\{i, i+1\}, \{i+1, i+2\}, \{i+2, *\}, \{*, i\}\}$, four circuits of the form $C_{(i,i+1)} := \{\{i, i+1\}, \{i+1, *\}, \{*, i\}\}$, and the fully symmetric circuit $C := \{\{1, 2\}, \{2, 3\}, \{3, 4\}, \{4, 1\}\}$ as well as the empty cycle and the two non-elementary cycles $C_{(1,2)} \cup C_{(3,4)}$ and $C_{(2,3)} \cup C_{(4,1)}$. Thus, the identities $C_{i+1} \oplus C_{i+2} = C_{(i,i+2)}$, $C_i \oplus C_{i+1} \oplus C_{i+2} = C_{(i+2,i+3)}$, and $C_1 \oplus C_2 \oplus C_3 \oplus C_4 = C$ imply that $\mathcal{B}^{(4)}$ is cyclically robust. Moreover, the fact that $C_1 \oplus C_2$ is a circuit, but $C_1 \cap C_2$ is not a circuit, implies that $\mathcal{B}^{(4)}$ is not strictly robust.

And yet another example of a graph with a cyclically robust cycle basis of G that is neither fundamental nor strictly robust is given in Figure 1 below:

We conjecture that even strictly robust cycle bases are not necessarily fundamental. Figure 2 below demonstrates the weaker claim that robust cycle bases are not necessarily strictly fundamental.

This example shows, furthermore, that non-Kirchhoff bases exists already for $\mu(G) = 3$.

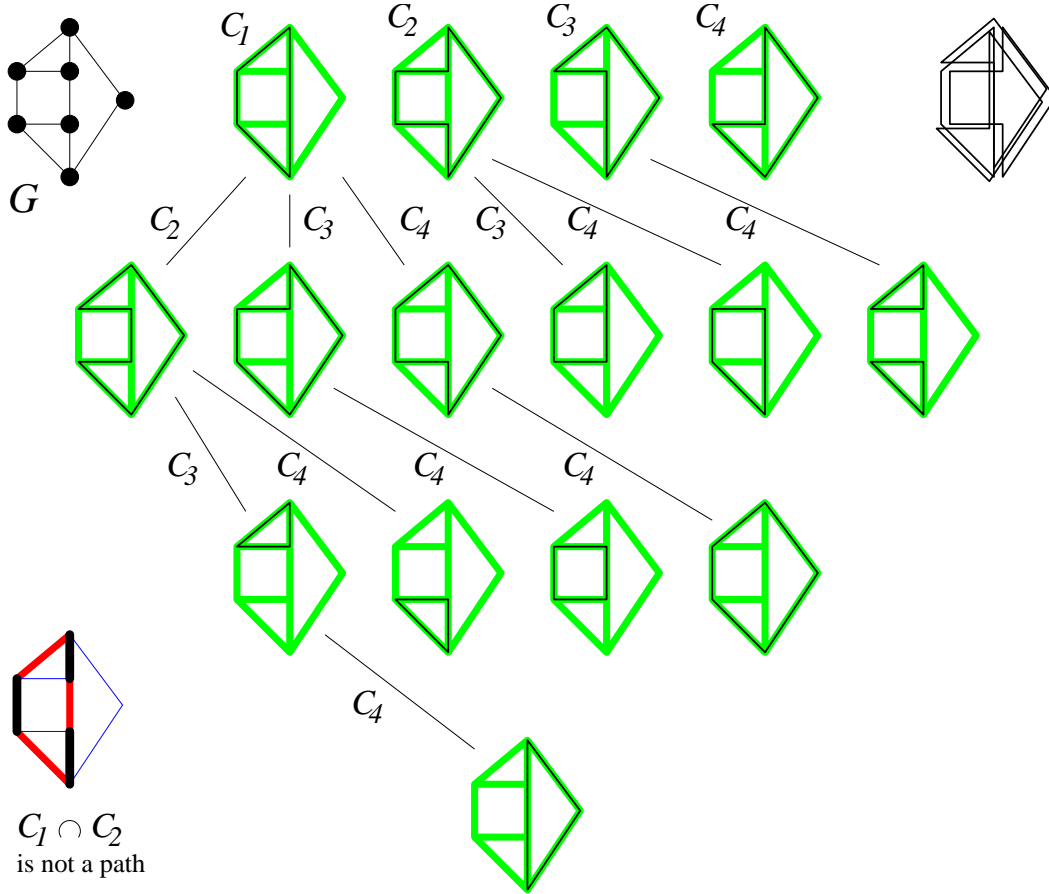


Fig. 1. $\mathcal{B} = \{C_1, C_2, C_3, C_4\}$ is a cycle basis of G that is cyclically robust, but neither fundamental nor strictly robust: As G is planar and has 4 bounded faces, we have $\mu(G) = 4$. The circuits in \mathcal{B} span the remaining 10 circuits of G as well as the single cycle $C_3 \oplus C_4$ that is not a circuit. The figure displays the graph $\Gamma_{\mathcal{B}}$ extended by a vertex representing the cycle $C_3 \oplus C_4$. Each vertex is represented by the cycle that defines it. And a sufficiently large collection of edges is labeled by the basis circuits that are to be added to go from one vertex to the next one. We see that \mathcal{B} is cyclically robust. Further, the top-right-most panel shows that the four basis cycles cover each edge of G at least twice. So, it cannot be fundamental.

It is easy to construct 2-connected graphs with cycle bases that are neither robust nor fundamental. To this end, let G' and G'' be 2-connected graphs with bases \mathcal{B}' and \mathcal{B}'' , resp. Suppose \mathcal{B}' is not fundamental and \mathcal{B}'' is not cyclically robust. We glue G' and G'' together by identifying two arbitrarily chosen edges e' from G' and e'' from G'' . Obviously, the resulting graph G has the basis $\mathcal{B} = \mathcal{B}' \cup \mathcal{B}''$. Since every cycle within G' (G'') is a linear combination of basis cycles from \mathcal{B}' (\mathcal{B}'') only, it is clear that \mathcal{B} is neither fundamental nor cyclically robust.

In [8] further examples can be found that demonstrate that (strictly) robust

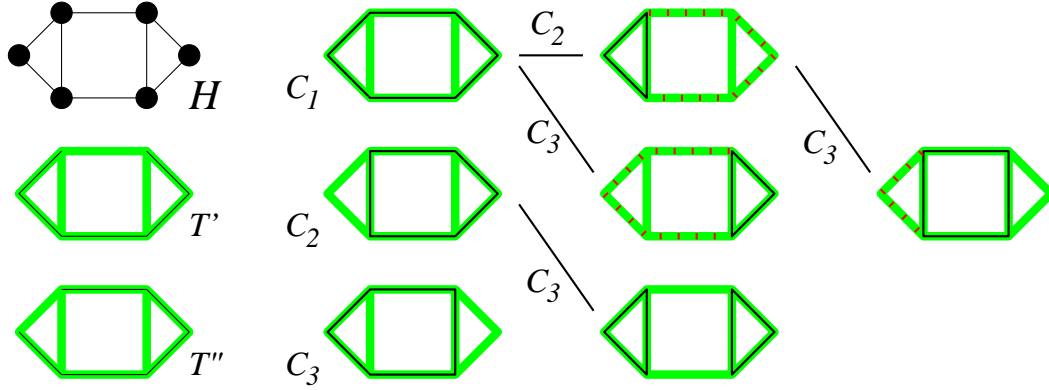


Fig. 2. $\mathcal{B} = \{C_1, C_2, C_3\}$ is a basis of the graph H . It is fundamental in view of $\mu(H) = 3$ and strictly robust (as the intersections $C_1 \cap C_2$ and $C_1 \cap C_3$ are paths), but not strictly fundamental (as C_1 is contained in $C_2 \cup C_3$).

and (strictly) fundamental are essentially unrelated concepts. These examples are compiled in Fig. 3 to make this contribution more self-contained.

Remark 1 *Note that, originally, “strictly well-arranged cycle sequences” were just called “well-arranged cycle sequences” and “strictly robust cycle bases” just “robust cycle bases” having the unfortunate consequence that the more restrictive notions sounded like being less restrictive. For that reason, we prefer the new nomenclature proposed above.*

5 A class of 3-regular graphs

Recall that a cycle C is elementary if and only if (1) C is connected *and* (2) all vertices of C have degree 2. For graphs whose degree is bounded by 3, the second condition is fulfilled for any generalized cycle: With degrees being even and bounded by 3, all vertices involved in C must have degree 2.

Here, we consider a particular class of Hamiltonian 3-regular graphs constructed as follows. For $m \geq 3$, let G_m denote the graph with the vertex set $V_m = \{1, 2, 3, \dots, 2m\}$ whose edge set E_m is the union of a Hamiltonian cycle and chords connecting “opposite” vertices: $E_m = R_m \cup D_m$ where $R_m = \{\{1, 2\}, \{2, 3\}, \dots, \{2m-1, 2m\}, \{2m, 1\}\}$ and $D_m = \{\{1, m+1\}, \{2, m+2\}, \dots, \{m, 2m\}\}$. Figure 4(a) shows $G_5 = (V_5, E_5)$.

Note that G_3 is the complete bipartite graph on $3 + 3$ vertices. For $m > 3$, $K_{3,3}$ is therefore a minor of G_m . Hence, G_m is non-planar for all even $n \geq 6$. Clearly, G_m is neither a complete graph nor a complete bipartite graph for

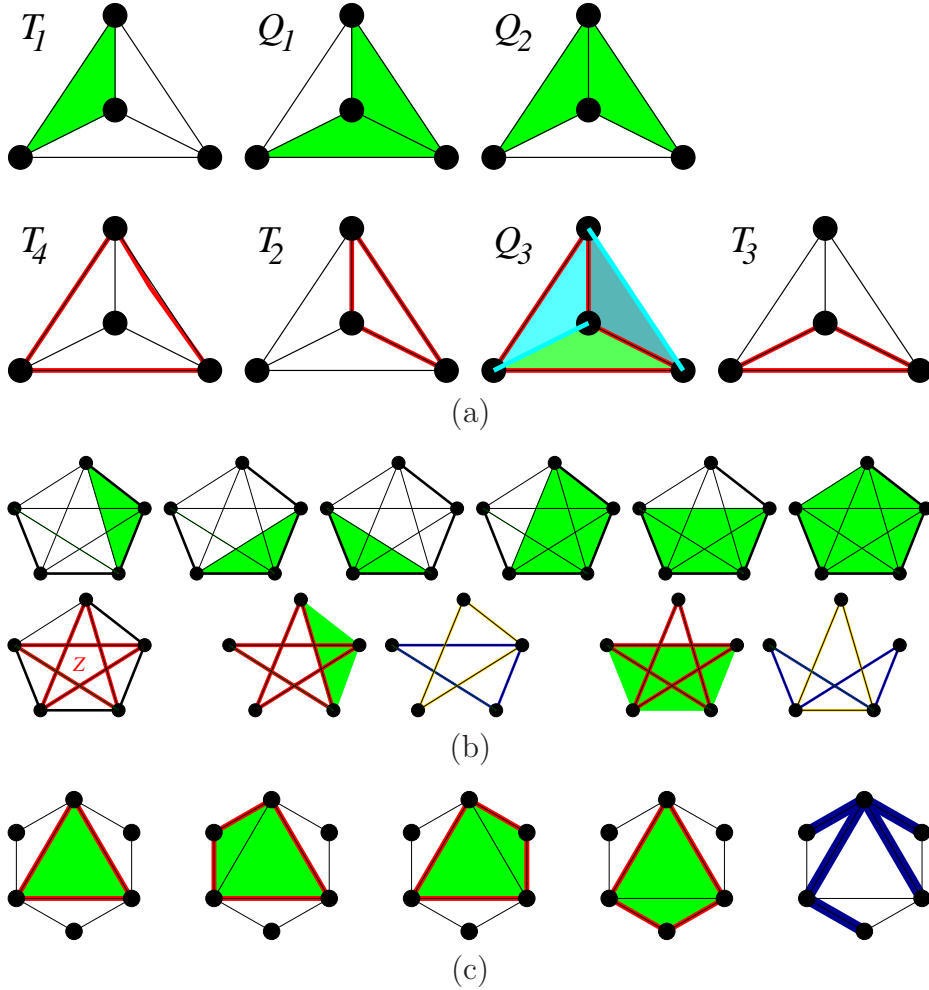


Fig. 3. Counter-examples discussed in Kainen's paper [8].
 (a) The basis $\mathcal{B} = \{T_1, Q_1, Q_2\}$ of K_4 is cyclically robust but not strictly robust: The four cycles that are not elements of the basis are obtained $T_4 = T_1 \oplus Q_1$, $T_2 = T_1 \oplus Q_2$, $Q_3 = Q_1 \oplus Q_2$, and $T_3 = T_1 \oplus Q_3 = T_1 \oplus (Q_1 \oplus Q_2)$. Hence \mathcal{B} is cyclically robust. However, $Q_1 \cap Q_2$ is not a path.
 (b) Ostrowski's example. The path P_5 is a spanning tree of K_5 . The corresponding basis \mathcal{B} is formed by the three triangles, the two quadrangles, and the pentagon shown in the upper row. The "inner" pentagon Z is the sum of the three triangles and the two quadrangles. The last cycle of its cycle sequence is either a triangle or a quadrangle. Because of symmetry, all three triangles and both quadrangles must be added to graph that are isomorphic to the two cases shown in the lower row. The \oplus -sum of the first four basis cycles, is not an elementary cycle but an edge-disjoint union of cycles in both cases. Thus \mathbf{B}_Z cannot be cyclically well-arranged.
 (c) In Vogt's example, the graph G is formed by a six-cycle with an inscribed triangle I . The cycle basis \mathcal{B} containing I and the three quadrangles is fundamental: Each quadrangle contains two edges that are not in any of the other basis cycles. A spanning tree T generating \mathcal{B} would have to contain exactly two edges from the inner triangle and at least one edge of each outer node of degree two. But then T would generate outer triangles, which do not belong to \mathcal{B} . Hence \mathcal{B} is not strictly fundamental. The outer six-cycle is the sum of the three quadrangles but the sum of any two quadrangles is not elementary which shows that \mathcal{B} is not cyclically robust.

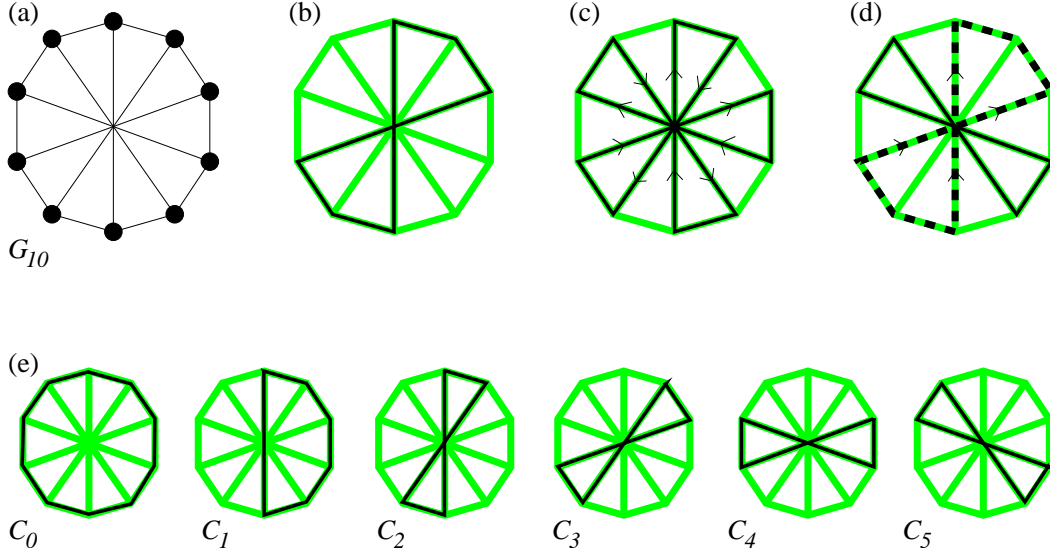


Fig. 4. (a) The graph G_5 defined in the text. (b) Cycles containing two chords of the outer cycle are elementary. (c) A circuit with five chords of the the outer cycle. (d) A disconnected cycle. (e) The robust cycle basis \mathcal{B}_5 defined in the text.

any $n \geq 6$. Thus G_m , $n > 6$ does not belong to the classes of graphs for which cyclically robust cycle bases have been shown to exist. In the remainder of this section we show that G_m has a cyclically robust cycle basis for all even $m \geq 3$.

By $(x \rightarrow z)$ we denote a path in R_m between end-vertices x and z . If $x < z$ the path $(x \rightarrow z)$ passes through all vertices y with $x < y < z$. If $x > z$ the path $(x \rightarrow z)$ passes through all vertices y with $x < y$ or $y < z$.

We start by characterizing the set of circuits of G_m . For a cycle C , consider the number d of edges of D_m it contains, $d = |C \cap D_m|$. In case $d = 0$, C is empty or $C = R_m$, hence C is elementary. Also for $d = 1$, C is elementary because C consists of an edge $\{x, x+m\} \in D_m$ and a path in R_m connecting x and $x+m$. For $d = 2$, we find two distinct edges $\{x, x+m\}, \{y, y+m\} \in C \cap D_m$, connected either by paths $(x \rightarrow y)$ and $(x+m \rightarrow y+m)$ or by paths $(y+m \rightarrow x)$ and $(x+m \rightarrow y)$ and, thus, forms a circuit as illustrated in Figure 4. Cycles with larger d , as illustrated in Figure 4(c,d), are captured by the following

Lemma 2 *Let C be a cycle on G_m with $d = |C \cap D_m| \geq 3$. Then, C is elementary if and only if d is odd.*

Proof. Let $r_1, r_2, r_3, \dots, r_{2d}$ be the pairwise distinct end vertices of the edges in $C \cap D_m$ in ascending order, $r_1 < r_2 < \dots < r_{2d}$, where $\{r_i, r_{d+i}\}$ is an edge in D_m for $1 \leq i \leq d$. For all $1 \leq i \leq 2d$, the degree of r_i is 2 implying that r_i is an end-vertex of a path in $C \cap R_m$ connecting r_i with either r_{i+1} or r_{i-1} (indices

computed modulo $2m$). Without loss of generality (after cyclically relabeling the vertices), C contains paths $(r_1 \rightarrow r_2)$, $(r_3 \rightarrow r_4)$, \dots , $(r_{2d-1} \rightarrow r_{2d})$. In particular, if $d \geq 3$ is even, C contains the paths $(r_1 \rightarrow r_2)$ and $(r_{d+1} \rightarrow r_{d+2})$ and the edges $\{r_1, r_{d+1}\}$, $\{r_2, r_{d+2}\}$ that form a circuit. The paths $(r_3 \rightarrow r_4)$ and $(r_{d+3} \rightarrow r_{d+4})$ and the edges $\{r_3, r_{d+3}\}$, $\{r_4, r_{d+4}\}$ are also contained in C and form another circuit. Having at least two disjoint circuits as subsets, C itself is not elementary. For the case of odd d , consider a walk along the cycle C : $(r_1 \rightarrow r_2)$, $\{r_2, r_{d+2}\}$, $(r_{d+2} \rightarrow r_{d+3})$, $\{r_{d+3}, r_3\}$, $(r_3 \rightarrow r_4)$, $\{r_4, r_{d+4}\}$, $(r_{d+4} \rightarrow r_{d+5})$, $\{r_{d+5}, r_5\}$, \dots , $(r_d \rightarrow r_{d+1})$, $\{r_{d+1}, r_1\}$. The walk starts and ends in r_1 and covers the whole cycle C . Being a closed path, C is elementary. \square

Define the set of cycles $\mathcal{B}_m = \{C_0, C_1, \dots, C_m\}$ with $C_0 = R_m$, $C_1 = (1 \rightarrow m+1) \cup \{1, m+1\}$, and C_i the 4-cycle induced by the vertices $\{i-1, i, i+m-1, i+m\}$ for $2 \leq i \leq m$, see Figure 4(a). The set \mathcal{B}_m is strictly fundamental: For $1 \leq i \leq m$, C_i contains the edge $\{i, m+1\}$ that is not in C_j for any $j < i$.

Proposition 1 \mathcal{B}_m is a cyclically robust cycle basis of G_m .

Proof. Consider the Kirchhoff basis \mathcal{A} generated by the spanning tree $(1 \rightarrow n)$. \mathcal{B}_m and \mathcal{A} contain the same number of elements $\mu(G_m) = m+1$. In order to prove that \mathcal{B}_m is a cycle basis we show that every cycle in \mathcal{A} is a sum of cycles in \mathcal{B}_m . The outer Hamilton cycle $R_m = C_0$ is contained in both \mathcal{A} and \mathcal{B}_m . The other m cycles in \mathcal{A} are of the form $\{i, i+m\} \cup (i \rightarrow i+m)$ for $1 \leq i \leq m$, which equals $\bigoplus_{j=1}^i C_j$ and thus is a sum over cycles in \mathcal{B}_m .

Consider a cycle $Q \in \text{span}\{C_0, C_1, \dots, C_{i-1}\}$ with $2 \leq i < m$. The addition of the “next” basis cycle C_i to Q leaves the number of edges out of D_m unchanged or increases it by 2, that is $|(Q \oplus C_i) \cap D_m| - |Q \cap D_m| \in \{0, 2\}$. This difference is zero, if Q contains the edge $\{i-1, i+m-1\}$ which cancels out under addition of C_i . Otherwise, the difference is 2 because C_i contributes 2 additional edges $\{i-1, i+m-1\}$ and $\{i, m+1\}$ to the sum cycle $Q \oplus C_i$.

Now let C be a circuit in G_m and find coefficients $\lambda_i \in \{0, 1\}$ such that $C = \bigoplus_{i=0}^m \lambda_i C_i$. The j -th partial sum is $Q_j = \bigoplus_{i=0}^j \lambda_i C_i$. The number of edges from D_m in the partial sum, $d_j := |Q_j \cap D_m|$, grows monotonically with j . For $j \geq 2$, the difference $d_j - d_{j-1}$ only takes values 0 and 2. Thus, if $|C \cap D_m| \leq 2$, then $|Q_j \cap D_m| \leq 2$ for all $0 \leq j \leq m$, so Q_j is elementary. If $d_m \geq 3$ then d_m is odd by Lemma 2 using that C is elementary. For all $j \geq 2$, the difference $d_j - d_{j-1}$ is either 0 or 2, so d_j is odd for all $j \geq 1$. With Lemma 2 we find that Q_j is elementary for $j \geq 1$. Also Q_0 , being either empty or one of the basis cycles in \mathcal{B}_m , is elementary. Each partial sum Q_i of the sum-sequence generating C is a circuit. \square

We finally conjecture that Hamilton graphs with vertex degree bounded by 3 have a cyclically robust cycle basis.

6 Concluding Remarks

In Fig. 5 we summarize the (lack of) mutual relationship between (strictly) fundamental and (strictly) robust cycle bases. Some of the counter-examples are sketched in Fig. 3.

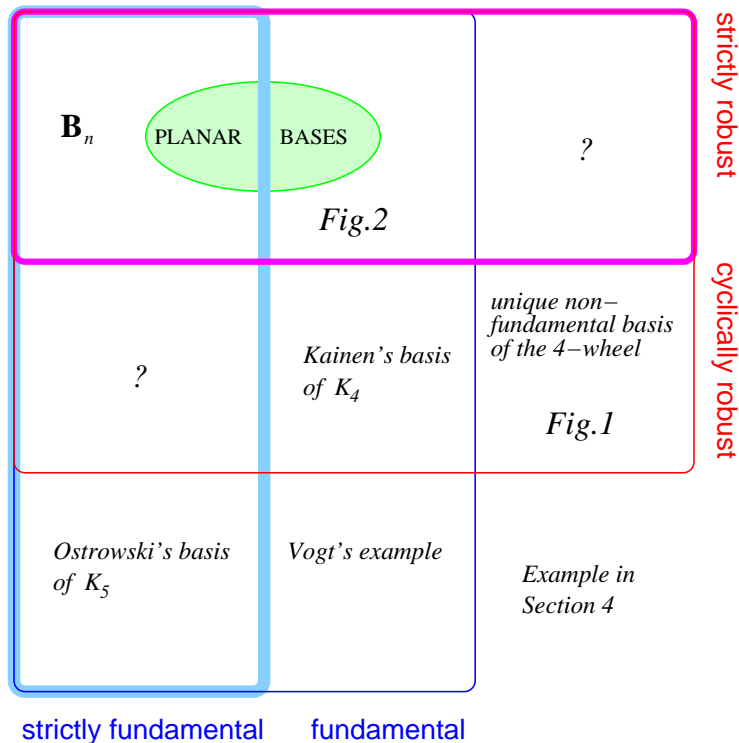


Fig. 5. Venn diagram of robust and fundamental bases. \mathbf{B}_n is the basis of K_n consisting of all triangles that contain vertex 1. Clearly \mathbf{B}_n is strictly fundamental, deriving from the star with central vertex 1, and robust as shown in [8]. Planar bases are robust and fundamental. Note that neither \mathbf{B}_n , $n > 4$ nor the basis in Fig. 2 are planar bases. Bases that are neither fundamental nor cyclically robust are discussed at the end of section 4. Three counter examples, Kainen's basis of K_4 , Ostrowski's basis of K_5 , and Vogt's example on $C_6 \cap C_3$ are taken from [8], sketched in Fig. 3. Additional examples of strictly fundamental bases that are not cyclically robust can be found in [4, 1]. We conjecture that examples also exist for the two combinations marked by “?”.

As discussed in the introductory section, cyclically robust bases are of utmost practical value for network analysis because they can be used to design computationally efficient and accurate Monte Carlo sampling procedures for analysing the cycle distribution in large networks. We have shown here that cyclic robustness is unrelated to (strictly) fundamental bases. Therefore, classical graph-theoretic constructions based on spanning trees or ear decompositions can at best help us to find subclasses of robust bases. We gave here an

example of a special class of cubic graphs where such an approach is successful and cyclically robust fundamental bases can be constructed.

We remark, finally, that the notions of (strictly) fundamental bases are of interest in the context of matroid theory in general. It is not clear, however, whether (cyclic) robustness also has a counterpart in this much more general setting, since the present definition of the latter concepts explicitly makes use of graph-theoretic concepts such as that of a “path”.

Acknowledgments

We thank Andreas Dress for valuable comments on this manuscript.

References

- [1] F. Berger, C. Flamm, P. M. Gleiss, J. Leydold, and P. F. Stadler. Counterexamples in chemical ring perception. *J. Chem. Inf. Comput. Sci.*, 44:323–331, 2004.
- [2] W.-K. Chen. On vector spaces associated with a graph. *SIAM J. Appl. Math.*, 20:525–529, 1971.
- [3] E. T. Dixon and S. E. Goodman. An algorithm for the longest cycle problem. *Networks*, 6:139–146, 1976.
- [4] U. Doğrusöz and M. Krishnamoorthy. Enumerating all cycles of a planar graph. *Journal of Parallel Algorithms and Applications*, 10:21–36, 1996.
- [5] L. Euler. Elementa doctrinae solidorum. *Novi Comm. Acad. Sci. Imp. Petropol.*, 4:109–140, 1752. (Latin).
- [6] D. Hartvigsen and R. Mardon. Cycle bases from orderings and coverings. *Discr. Math.*, 94:81–94, 1991.
- [7] D. Hartvigsen and E. Zemel. Is every cycle basis fundamental? *J. Graph Theory*, 13:117–137, 1989.
- [8] P. C. Kainen. On robust cycle bases. In Y. Alavi, D. Jones, D. R. Lick, and J. Liu, editors, *Electronic Notes in Discrete Mathematics*, volume 11. Elsevier Science Publishers, 2002. Proceedings of the Ninth Quadrennial International Conference on Graph Theory, Combinatorics, Algorithms and Applications; June 4-9 2000, Kalamazoo, MI.
- [9] G. Kirchhoff. Über die Auflösung der Gleichungen, auf welche man bei der Untersuchungen der linearen Verteilung galvanischer Ströme geführt wird. *Poggendorf Ann. Phys. Chem.*, 72:497–508, 1847.
- [10] K. Klemm and P. F. Stadler. Statistics of cycles in large networks. *Phys. Rev. E*, 73:025101, 2006.
- [11] C. Liebchen and R. Rizzi. Classes of cycle bases. *Discr. Appl. Math.*, 155:337–355, 2007.

- [12] M. M. Sysło. Characterizations of outerplanar graphs. *Discrete Math.*, 26:47–53, 1979.
- [13] M. M. Sysło. An efficient cycle vector space algorithm for listing all cycles of a planar graph. *SIAM J. Comput.*, 10:797–808, 1981.
- [14] A. Wagner and J. Wright. Alternative routes and mutational robustness in complex regulatory networks. *BioSystems*, 88:163–172, 2007.
- [15] H. Whitney. Nonseparable and planar graphs. *Trans. Amer. Math. Soc.*, 34:339–362, 1932.
- [16] H. Whitney. On abstract properties of linear dependence. *Am. J. Math.*, 57:509–533, 1935.