

On the Synchronization of Networks with Prescribed Degree Distributions

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Abstract

We show that the degree distributions of graphs do not suffice to characterize the synchronization of systems evolving on them. We prove that, for any given degree sequence satisfying certain conditions, a connected graph having that degree sequence exists for which the first nontrivial eigenvalue of the graph Laplacian is arbitrarily close to zero. Consequently, dynamical systems defined on such graphs have poor synchronization properties. The result holds under quite mild assumptions, and shows that there exists classes of random, scale-free, regular, small-world, and other common network architectures which impede synchronization. The proof is based on a construction that also serves as an algorithm for building non-synchronizing networks having a prescribed degree distribution.

Index Terms—Synchronization, networks, graph theory, Laplacian

1 Introduction

Many network architectures encountered in nature or used in applications are described by their degree distributions. In other words, if $P(d)$ denotes the fraction of vertices having d incident edges, then the shape of the function $P(d)$ distinguishes certain network classes. For example, in the classical random graphs studied by Erdős and Rényi [1, 2], $P(d)$ has a binomial distribution, which converges to a Poisson distribution for large network sizes. The degree distribution of regular networks, where every vertex has the same degree k , are given by the delta function $P(d) = \delta(d - k)$, whose small perturbations by random rewiring introduce small-world effects [3, 4]. Many common networks, such as the World-Wide Web [5], the Internet [6], and, networks of protein interactions [7] have

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been shown to have approximate power-law degree distributions, and have been termed *scale-free* [8]. The power grid has exponentially distributed vertex degrees [3]. Some social networks have distributions similar to a power-law, possibly with some deviations or truncations at the tails [9]. A recent survey is given in [10]. These examples demonstrate the recent widespread effort in classifying common large networks according to their degree distributions.

In many applications, the vertices of networks have internal dynamics, and one is interested in the time evolution of a dynamical system defined on the network. A natural question is then what, if anything, the degree distribution can say about the dynamics on the network. Asked differently, to what extent does a classification according to degree distributions reflect itself in a classification according to different qualitative dynamics? The question is all the more significant since many large networks are constructed randomly. Hence, in essence every realization of, say, a power-law distribution is a different network. On the one hand, it is conceivable that the dynamics on different realizations could be sufficiently different. On the other hand, it is known that these realizations can have certain common characteristics; for instance, small average distances and high local clustering found in small-world networks [10]. Consequently, the relation between the degree distribution and the dynamics is not trivial. The present work studies this relation in the context of a specific but important dynamical behavior of networked systems, namely synchronization.

It is well-known that the synchronization of diffusively-coupled systems on networks is crucially affected by the network topology. In particular, the so-called spectral gap, or the smallest nontrivial eigenvalue λ_1 , of a discrete Laplacian operator plays a decisive role for chaotic synchronization: Larger values of λ_1 enable synchronization for a wider range of parameter values, in both discrete and continuous-time systems [11–13], and also in the presence of transmission delays [14]. Here we shall prove that, given any degree distribution satisfying certain mild assumptions, a connected network having that distribution can be constructed in such a way that λ_1 is inversely proportional to the number of edges in the graph. Hence, there exist large networks with these degree distributions which are arbitrarily poor synchronizers. The proof is based on ideas from degree sequences of graph theory, and in particular makes use of the relation between λ_1 and the Cheeger constant of the graph. It applies to the class of distributions that include the classical random (Poisson), exponential, and power-law distributions, as well as nearest-neighbor-coupled networks and their small-world variants, and many others. Thus, we establish that the degree distribution of a network does not suffice to characterize the synchronizability of the dynamics on its vertices.

Our proof is constructive in nature. Hence, from another perspective, it can be viewed as an algorithm for designing non-synchronizing networks having a prescribed degree distribution. This may have implications for engineering systems which should essentially operate in asynchrony, such as neuronal networks. Furthermore, the non-synchronizing networks that we construct can actually have quite small diameters or average distances. This proves that the informal arguments which refer to efficient information transmission via small distances as a mechanism for synchronization are ill-founded. For a related dis-

discussion based on numerical studies of small-world networks, see [12, 15]. Here we provide a rigorous proof for a much larger class of distributions.

2 Degree distributions

In the following we consider finite graphs without loops or multiple edges. As mentioned above, the degree distribution $P(d)$ gives the fraction of vertices having d incident edges. A related notion is that of a *degree sequence*, which is a list of nonnegative integers $\pi = (d_1, \dots, d_n)$ where d_i is the degree of the i th vertex. For each graph such a list is well-defined, but not every list of integers corresponds to a graph. A sequence $\pi = (d_1, \dots, d_n)$ of nonnegative integers is called *graphic* if there exists a graph G with n vertices for which d_1, \dots, d_n are the degrees of its vertices. G is then referred to as a *realization* of π . A characterization of graphic degree sequences is given by the following.

Lemma 1 ([16, 17]) *For $n > 1$, the nonnegative integer sequence π with n elements is graphic if and only if π' is graphic, where π' is the sequence with $n - 1$ elements obtained from π by deleting its largest element d_{\max} and subtracting 1 from its d_{\max} next largest elements. The only 1-element graphic sequence is $\pi_1 = (0)$.*

Often one is interested in connected graphs, and the following result gives the correspondence between degree sequences and connected realizations (see e.g. [18]).

Lemma 2 *A graphic sequence π with n elements has a connected realization if and only if the smallest element of π is positive and the sum of the elements of π is greater or equal than $2(n - 1)$.*

One of the simplest degree sequences belong to k -regular graphs, in which each vertex has precisely k neighbors. Here, the degree sequence is given by

$$\pi = (k, \dots, k). \tag{1}$$

There is a simple criterion for such constant sequences to be graphic (see e.g. [18]).

Lemma 3 *A sequence $\pi = (k, \dots, k)$ with n elements is graphic if and only if n or k is even.*

The construction of *small-world* networks starts with a k -regular graph with k much smaller than the graph size n , e.g. a large circular arrangement of vertices which are coupled to their near neighbors. Then a small number of edges c are added between randomly selected pairs of vertices [4]. When $c \ll n$, the degree of a vertex typically increases by at most one, which yields the degree sequence

$$\pi = (k + 1, \dots, k + 1, k, \dots, k) \tag{2}$$

where $2c$ vertices have degree $k + 1$. In a variant of the model [3], randomly selected edges are *replaced* by others, so the degree of a vertex may increase or decrease by one, and the degree sequence becomes

$$\pi = (k + 1, \dots, k + 1, k, \dots, k, k - 1, \dots, k - 1) \quad (3)$$

where the number of vertices having degree $k + 1$ or $k - 1$ are each equal to c . More generally, there is also the possibility of having some vertex degrees increase or decrease by more than one, in which case the sequences (2) and (3) are modified accordingly, while the sum of the degrees remain equal to $nk + 2c$ and nk , respectively.

In addition to these well-known graph types, we shall also consider more general sequences, which we define as follows.

Definition A sequence π with largest element d_{\max} is called a *full sequence* if each integer d satisfying $1 \leq d \leq d_{\max}$ is an element of π , and the sum of the elements of π is even.

We also give a criterion for full sequences to be graphic.

Lemma 4 *Let $\pi = (d_{\max}, \dots, d_{\max}, d_{\max} - 1, \dots, d_{\max} - 1, \dots, 1, \dots, 1)$ be a full sequence with n elements and $d_{\max} \leq n/2$. Then π is graphic.*

Proof. We prove by induction on the number of elements of π . This is trivial for the full sequence with two elements. Suppose that every full sequence with at most $n \geq 2$ elements and largest element not larger than $n/2$ is graphic. Let π be a full sequence with $n + 1$ elements and largest element $d_{\max} \leq (n + 1)/2$. We look at the sequence π' that is defined in Lemma 1. It is easy to see that π' is a full sequence. Let d'_{\max} be the largest element of π' . By the definition of π' , $d'_{\max} \leq d_{\max}$. We claim that

$$d'_{\max} \leq n/2. \quad (4)$$

For if $d'_{\max} > n/2$, then

$$n/2 < d'_{\max} \leq d_{\max} \leq (n + 1)/2.$$

This implies

$$d'_{\max} = d_{\max} = (n + 1)/2 \quad (5)$$

so the number of d_{\max} 's in π is at least $d_{\max} + 2$. Since π is a full sequence, the number of its elements is then

$$n + 1 \geq (d_{\max} + 2) + (d_{\max} - 1),$$

implying $d_{\max} \leq n/2$, which contradicts (5). Thus, (4) holds. Then π' is graphic by induction, and by Lemma 1 π is graphic. ■

The reason for introducing the concept of full sequences is that most graph types that are of interest have full degree sequences. For instance, large random graphs of Erdős-Rényi have their vertex degrees distributed according to the Poisson distribution

$$P(d) \sim \frac{\mu^d e^{-\mu}}{d!}, \quad d \geq 1. \quad (6)$$

Since such networks are randomly constructed, (6) is understood to hold in the limit as the network size increases while μ is kept constant. In scale-free networks, the degree distribution follows a power law

$$P(d) \sim d^{-\beta} \quad (7)$$

for some $\beta > 1$. The exponential distribution obeys

$$P(d) \sim e^{-\mu d}$$

with $\mu > 0$. Of course, in any finite graph these distributions are truncated from the right since the maximum degree is finite, and then one has $P(d) > 0$ for $1 \leq d \leq d_{\max}$. Furthermore, the sum of the vertex degrees is always even since it is twice the number of edges in the graph. Therefore, finite graphs approximated by these distributions, or more generally by any distribution satisfying $P(d) > 0$ for $1 \leq d \leq d_{\max}$, have a full degree sequence. In the next section, we prove that full degree sequences, as well as the regular and small-world networks (1)-(3), have a realization which is a poor synchronizer. Furthermore, the synchronizability of this realization worsens with increasing graph size.

3 Constructing networks with small spectral gap

The synchronization of dynamical systems is generally studied in diffusively coupled equations, which in discrete-time may have the form

$$x_i(t+1) = f(x_i(t)) + \kappa \left[\frac{1}{d_i} \sum_{\substack{j \\ j \sim i}} f(x_j(t)) - f(x_i(t)) \right], \quad i, \dots, n. \quad (8)$$

In (8), x_i denotes the state of the i th unit, which is viewed as a vertex of a graph, and has d_i neighbors to which it is coupled. The notation $i \sim j$ denotes that units i and j are coupled, which is represented in the underlying graph by an edge, and κ quantifies the coupling strength. The system is said to synchronize if $x_i(t) \rightarrow \bar{x}(t)$ as $t \rightarrow \infty$ for all i , where $\bar{x}(t)$ is called the synchronized solution. Systems similar to (8) arise in different applications and coupling types, and also in continuous time. In all cases, the effect of the network topology on the stability of the synchronized solution is determined by the eigenvalues of a coupling operator. For (8), the relevant operator is the (normalized) Laplacian with matrix representation

$$L = D^{-1}A - I \quad (9)$$

where $D = \text{diag}\{d_1, \dots, d_n\}$ is the diagonal matrix of vertex degrees and $A = [a_{ij}]$ is the adjacency matrix of the graph, defined by $a_{ij} = 1$ if $i \sim j$ and zero otherwise. The eigenvalues of L are real and nonpositive, which we denote by $-\lambda_i$. Zero is always an eigenvalue, and has multiplicity 1 for a connected graph. The smallest nontrivial eigenvalue, denoted λ_1 , is called the spectral gap of the Laplacian, and is the important quantity for the synchronization of coupled chaotic systems. Larger values of λ_1 enable chaotic

synchronization for a larger set of parameter values. This result holds in both continuous and discrete time, for the Laplacian (9) that we consider here, as well as the combinatorial Laplacian $A - D$ [11–13]. In the following, we shall show how to construct a graph having a prescribed degree sequence and an arbitrarily small spectral gap, that is, a poor synchronizer.

We denote the cardinality of the set S by $|S|$. Let $G = (E, V)$ be a connected graph, with edge set E and vertex set V . For a subset $S \subset V$, we define

$$h_G(S) = \frac{|E(S, V - S)|}{\min(\sum_{v \in S} d_v, \sum_{u \in V - S} d_u)}, \quad (10)$$

where $|E(S, V - S)|$ denotes the number of edges with one endpoint in S and one endpoint in $V - S$. The *Cheeger constant* h_G is defined as

$$h_G = \min_{S \subset V} h(S).$$

This quantity provides an upper bound for the smallest nontrivial eigenvalue.

Lemma 5 ([19]) $\lambda_1 \leq 2h_G$.

Based on this estimate we construct, from a given degree distribution, a realization whose spectral gap λ_1 is small. We first consider regular graphs.

Theorem 1 Suppose $\pi = (k, \dots, k)$ is graphic with n elements, with $2 \leq k < n/2$. Then π has a connected realization G such that

$$\lambda_1(G) \leq \frac{4}{|E(G)| - k}.$$

Proof. We split π in two sequences π_1 and π_2 , which are either equal or π_1 has one element less than π_2 . By Lemmas 2 and 3, π_1 and π_2 have connected realizations G_1 and G_2 , respectively. Now we construct a connected realization G of π as follows (see Figure 1). Let uv and xy be edges of G_1 and G_2 that are in a cycle. We delete uv and xy , and add new edges ux and vy . This new graph G is connected, and it is a realization of π since vertex degrees remain unchanged after this operation. Furthermore, if S denotes the smaller of the vertex sets $V(G_1)$ and $V(G_2)$, we have

$$h_G(S) = \frac{2}{\sum_{v \in S} d_v} \leq \frac{2}{nk - k} = \frac{2}{|E(G)| - k} \quad (11)$$

Thus the Cheeger constant of G satisfies $h_G \leq 2/(|E(G)| - k)$, and by Lemma 5, $\lambda_1(G) \leq 4/(|E(G)| - k)$. ■

Next we consider the small-world variants of regular graphs.

Theorem 2 Let π be a k -regular degree sequence with n elements, with $k < n/2$, and let π' be a small-world degree sequence obtained from π by adding or replacing c edges. Then π' has a connected realization G such that

$$\lambda_1(G) \leq \frac{2(c + 2)}{|E(G)| - (k + c)}.$$

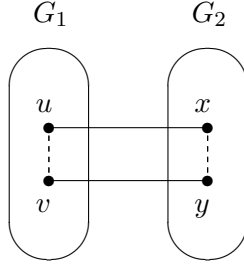


Figure 1: Partitioning the degree sequence π in two parts π_1 and π_2 , where G_1 and G_2 are realizations of π_1 and π_2 , respectively.

Proof. First we split the k -regular degree sequence π in two parts, as in the previous proof, obtaining the situation shown in Figure 1, and obtain the estimate (11) for $h_G(S)$, where S is the smaller of the vertex sets $V(G_1)$ and $V(G_2)$. Now we add or replace c edges in G . The numerator in (11) can then increase by at most c , in case the c added edges are all between G_1 and G_2 . Furthermore, the denominator can decrease by at most c , in case all the removed edges, if any, are all in G_1 . Hence,

$$h_G(S) \leq \frac{2 + c}{|E(G)| - k - c}$$

and the result follows by Lemma 5 as before. ■

Remark By the same argument, a connected graph G can be constructed from a k -regular graph by *removing* c edges, whose spectral gap satisfies

$$\lambda_1(G) \leq \frac{4}{|E(G)| - (k + c)}.$$

The method of proof for the above results is actually applicable to a large class of degree sequences, including the full sequences defined in Section 2.

Theorem 3 *Let $\pi = (d_{\max}, \dots, d_{\max}, d_{\max} - 1, \dots, d_{\max} - 1, \dots, 1, \dots, 1)$ be a full graphic sequence with n elements, $d_{\max} \leq n/4$, for which the sum of the elements is greater or equal than $2n + d_{\max}$, and each k , $1 \leq k \leq d_{\max}$, appears more than once. Then π has a connected realization G such that*

$$\lambda_1(G) \leq \frac{4}{|E(G)| - d_{\max}/2}.$$

Proof. For $k = 1, \dots, d_{\max}$, let n_k be the number of times the element k appears in the sequence π . We construct two sequences π_1 and π_2 from π as follows. The sequences π_1 and π_2 get $\lfloor n_k/2 \rfloor$ elements from each k for $k = 1, \dots, d_{\max}$, where the notation $\lfloor x \rfloor$

denotes the largest integer less than or equal to x . Let d_1, \dots, d_s be the remaining elements of π , where the d_i are distinct. For the set $N = \{1, 2, \dots, s\}$, we define

$$r_\pi = \min_{J \subseteq N} \left| \sum_{j \in J} d_j - \sum_{j \notin J} d_j \right|.$$

By induction on the number of elements, it is easy to see that $r_\pi \leq d_{\max}$. Now we spread the remaining elements d_1, \dots, d_s to the sequences π_1 and π_2 such that r_π is minimum. By Lemmas 4 and 2, π_1 and π_2 have connected realizations G_1 and G_2 , respectively. Let uv and xy be edges of G_1 and G_2 that are in a cycle or a vertex of the edges has degree one. (Such edges exist since G_1 and G_2 are connected.) We reconnect these as before (Fig. 1), and the rest of the proof follows analogously to the proofs of Theorems 1 and 2. ■

We note that many of the assumptions in Theorems 1-3 can be relaxed. Furthermore, the estimates obtained for λ_1 are certainly not tight. Our aim here is not to obtain estimates in full generality, but rather show that the spectral gap cannot be determined from the degree distribution. So we content ourselves to constructing connected graphs for which λ_1 is inversely proportional to the number of edges, i.e. is arbitrarily small for large graph sizes. Thus, based on Theorems 1-3, we conclude that the degree distribution of a network in general is not sufficient to determine its synchronizability.

The construction used in the above proofs has a quite general nature. In essence, the idea is to split a given sequence π into two sequences π_1, π_2 more or less equal in size, which have the connected realizations G_1, G_2 , respectively. Then, as schematically shown in Figure 1, switching the edges between two pairs of vertices yields a connected graph G as a realization of π . The spectral gap λ_1 is estimated from the Cheeger constant of the sets G_1 or G_2 , and is small since the numerator in (10) is 2 and the denominator can be rather large. This gives an algorithm for constructing networks having a prescribed degree distribution and a small spectral gap. In certain cases, it may require some effort to obtain connected realizations for the partitions π_1, π_2 . Nevertheless, by arguments similar to the above, it is not hard to show that if π_1 has a realization with m components, then it is possible to construct a realization G of π such that λ_1 is essentially given by $\lambda_1 \sim 2m/|E(G)|$. So, the basic idea of the construction is indeed applicable to much more general cases and distributions than we have considered here.

4 Discussion and conclusion

We have shown that the degree distribution is not sufficient to characterize the synchronizability of a network. The method of proof is based on the construction of a graph which has a specified degree distribution and a small spectral gap for the graph Laplacian. The construction works for a wide class of degree distributions, including Poisson, exponential, and power-law distributions, regular and small-world networks, and essentially for any distribution P satisfying $P(d) > 0$ for $1 \leq d \leq d_{\max}$. Thus, synchronizability is not an intrinsic property of a degree distribution. Furthermore, small diameters or average distances in a

graph do not necessarily imply synchronization, since the poorly-synchronizing graphs we construct can have small diameters, as is clear from Figure 1.

The method of proof presented here is essentially an algorithm to construct a graph with a given degree distribution and a small spectral gap. Such a poor-synchronizing realization of a degree distribution is certainly not unique. An interesting query is then finding how many poor-synchronizers there are, as compared to all realizations of a given distribution? This turns out to be a difficult question. In general it is not easy to estimate of the number of realizations of a given degree sequence. McKay and Wormald [20] give the following estimate that depends on the maximum degree.

Lemma 6 ([20]) *Let $\pi = (d_1, \dots, d_n)$ be a degree sequence without zero elements, n_1 entries of value 1, and n_2 entries of value 2. If $n_1 = O(n^{1/3})$, $n_2 = O(n^{2/3})$, and $d_{\max} \leq \frac{1}{3} \log n / \log \log n$, then the number of the realizations of π is asymptotically*

$$\frac{M!}{(M/2)! 2^{M/2} d_1! \dots d_n! n!} \exp \left(-O(d_{\max}^6) - O\left(\frac{d_{\max}^{10}}{12n}\right) + O\left(\frac{d_{\max}^3}{M}\right) \right),$$

where $M = \sum d_i = 2|E(G)|$.

As an attempt to apply this result to the case considered here, let $\pi = (d_1, \dots, d_{2n})$ be a full degree sequence with $2n$ elements and $\pi_1 = (d_1, \dots, d_n)$ be one of the partitions with n elements such that $\sum_{i=1}^{2n} d_i = 2 \sum_{i=1}^n d_i$. Then the ratio of the number of realizations π_1 to π is approximately (by using $n! \approx \sqrt{2\pi n} \left(\frac{n}{e}\right)^n$)

$$d_{n+1}! \dots d_{2n}! n^n 2^{-M+2n} M^{-M/2} \exp \left(M/2 - n - O\left(\frac{d_{\max}^{10}}{12n}\right) \right),$$

where $M = \sum_{i=1}^n d_i = O(n \log n / \log \log n)$. For a certain partition π_1 the ratio is very small. On the other hand it is hard to give an estimate on the number of suitable partitions like π_1 that make λ_1 very small. Hence, at this point there seems to be no general estimates for the relative size of poor-synchronizers belonging to the same degree sequence, and the question remains open for future research. At any rate, it is clear that care is needed when making general statements about the synchronizability of, say, scale-free networks, even in a statistical or asymptotical sense. In this context it is also worth noting that most results about the scale-free architecture are based on the algorithm of Barabási and Albert [21], which, by way of construction, possibly introduces more structure to affect the dynamics than is reflected in the power-law degree distribution. In fact, another preferential attachment type algorithm introduced in [22] yields also graphs with power-law degree distributions, but significantly smaller first eigenvalues than those constructed by the algorithm of Barabási and Albert or random graphs. While the algorithm of [21] lets nodes receive new connections with a probability proportional to the number of connections they already possess, the one of [22] introduces new nodes by first connecting to an arbitrary node and then adding further connections that complete triangles, that is, to nearest neighbors of nodes already connected to. Since the probability to be a nearest node of another node is also proportional to the number of links a nodes possesses, that

algorithm also implements a preferential attachment scheme and therefore produces a power-law graph. The small first eigenvalues reported in [22] agree with our more general results in showing that it is possible to construct power-law graphs with arbitrarily small spectral gap. Furthermore, this observation holds regardless of the exponent in the power law.

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