Threshold Behavior and Aggregate Fluctuation

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Abstract

This paper concerns a propagation mechanism in an economy where many individuals follow a threshold rule and interact with a positive feedback. We derive an asymptotic distribution of the propagation size when the number of the agents tends to infinity. The propagation distribution exhibits a slower convergence to a deterministic value than it would if the agents followed a smooth adjustment policy. This gives rise to significant aggregate fluctuations in a finite lumpy-adjusting economy even when the agents are hit by small independent shocks. The result is applied to a standard sectoral business cycle model.

JEL classification: E1, E3

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1 Introduction

This paper presents a complete characterization of aggregate fluctuations when individual behavior follows a threshold rule, often called an (S, s) policy. Threshold rules are widely observed in individual economic behavior. As their microeconomic foundation has been well established, researchers' interest has shifted to their aggregate consequences. Many macroeconomic studies attempted to establish the aggregate relevance of the individual threshold behavior. For example, the menu-cost pricing model (Mankiw (1985)) claimed the aggregate effect of individual inertia in price settings. The model of lumpy investment (Cooper and Haltiwanger (1993)) claimed that the micro-level lumpy adjustment generated aggregate fluctuations in production.

Yet few theoretical works have shown the relevance of the threshold behavior in aggregation when the shocks are independent across individual units, an environment in which the law of large numbers takes effect. In the literature of the (S, s) economy, Caplin (1985) first raised the question of whether lumpy adjustments could cause significant fluctuations. His paper allowed individual shocks to be correlated but did not allow for positive feedback. Subsequent research, notably by Caplin and Spulber (1987) and Caballero and Engel (1991), studied the case where the feedback was present. They found that the mean aggregate behavior did not differ between the two economies, one with lumpy adjustments and the other with smooth adjustments. This neutrality result is due to the fact that adjustments in the extensive margin across agents work exactly like adjustments in the intensive margins within agents, when agents' positions in (S, s) bands are distributed uniformly. Caplin and Leahy (1997) showed that the uniform distribution is not obtained when an aggregate shock is present, in which case the adjustment in the extensive margin differs from that in the intensive margin.

This paper develops a mathematical method to evaluate the asymptotic distribution of aggregate variables when the number of agents N tends to infinity. Our approach clarifies when and why the lumpy adjustment matters. We reproduce the neutrality theorem that the mean aggregate behaviors of lumpy and smooth economies coincide, but we find that the variance of the propagation size in the lumpy economy converges to zero at a rate N-times slower than in the smooth economy. This raises the possibility that idiosyncratic shocks cause significantly large aggregate fluctuations for an economy with finite but many agents. Furthermore, the propagation size follows a heavy-tailed distribution when the idiosyncratic shocks are small relative to the size of the lumpiness, whereas it follows a normal distribution when the shock dominates the lumpiness. In the latter case, the aggregate behavior is the same as its smoothly-adjusting counterpart. The heavy-tailed distribution emerges from the propagation effect, which characterizes the aggregate fluctuations as long as the exogenous shocks do not overwhelm the feedback effect of individual lumpy adjustments.

Our model differs from Caplin and Spulber (1987) or Caballero and Engel (1991) in that we have a finite number of agents, whereas their models use a continuum of agents. Their neutrality theorem shows a deterministic (mean) property of the propagation, whereas our result shows a distribution of the propagation that converges to the deterministic mean at the infinite limit of the number of agents. By using a finite agent model, we can quantitatively evaluate how the distribution of propagation size relates to the number of agents. We also condition the size of the exogenous shocks on the number of agents, in order to assess for what magnitude of the shocks the aggregate fluctuation is determined by the feedback effect rather than by the exogenous shocks. Heuristically speaking, exogenous independent shocks across agents cause some agents to adjust. Their adjustments induce further adjustments of other agents through a feedback effect, which in turn induce even further adjustments, and so on. This chain reaction constitutes a propagation mechanism. If an economy is inhabited by a continuum of agents, then any positive shocks across agents cause a deterministic fraction of agents to adjust, due to the work of the law of large numbers at the limit. If the economy consists of countably many agents and the size of the shock is conditioned on the number of agents, then the outcome depends on how we condition on them. Suppose that the shock size is of order $1/N$. Then the *number* of agents that adjust due to the initial shocks asymptotically follows a non-degenerate Poisson distribution. The same mechanism applies to the subsequent propagation process when the feedback effect is of order $1/N$. The feedback effect is of order $1/N$ when the feedback is caused through an average behavior of all agents. In this case, the magnitude of the shocks is equal to the magnitude of the feedback effect, and the feedback effect plays a decisive role in determining the aggregate fluctuations. The *fraction* of agents that adjust converges to a deterministic zero[1], but the convergence rate can be slower than the law of large numbers predicts. We find that this is indeed the case. Thus, the finite agent model enables us to study the economic situations where individual shocks add up to a sizable aggregate shock, which the continuum of agents models could not account for.

The argument above implies that the asymptotic distribution of the aggregate variable qualitatively differs depending on the size of the shock. In fact, the size of lumpiness relative to the size of shocks determines the phase of the propagation distribution. We can regard the smooth and lumpy economies as polar cases in terms of the ratio of the lumpiness over the shock size. When the ratio goes to zero, the propagation follows a normal distribution and the aggregate variance in the lumpy economy is as

[1]By the statement "a random variable converges to a deterministic value," we mean that the variance of the random variable converges to zero and the mean converges to the deterministic value.

small as the smooth economy, whereas when the ratio diverges to infinity, it follows a heavy-tailed distribution and the variance in the lumpy economy is N-times bigger than the smooth economy. By numerically calculating the aggregate variances in the region between the polar cases, we determine that the transition between the two phases occurs at the point where the lumpiness is equal to the standard deviation of the shock. We also find that the heavy-tailed distribution is infinitely divisible. This property is useful when we consider a dynamic extension of the model. Suppose that the exogenous shocks hit repeatedly over time. Then, the variance of the accumulated shocks increases linearly with the time horizon. Thus, the corresponding aggregate fluctuation follows the heavy-tailed distribution when the time horizon is short, whereas it follows the normal distribution when the time horizon is long. Using the infinite divisibility, we can derive an approximate stochastic process of the aggregate for a short time horizon, which progressively turns into a normal process for a long time horizon.

As an application, our approach makes it possible to model the conjecture that business cycles are driven by the propagation of sectoral adjustments. The disaggregate view of business cycles also has historically encountered difficulties in overcoming the law of large numbers. The real business cycle model by Kydland and Prescott (1982) suffered the criticism that no aggregate technological shocks large enough to endorse their approach were observed (for example, Summers (1986)). A sectoral model by Long and Plosser (1983) showed that independent sectoral shocks could in principle generate comovement across sectors in a general equilibrium setting. However, the comovement was weaker than needed to regard the finely disaggregated sectoral shock as the fundamental shock to aggregate fluctuations. Dupor (1999) rigorously made the point that it was unlikely that independent sectoral shocks could provide an amplification mechanism stronger than the law of large numbers. An attempt to break the law of large numbers was made by Horvath (2000), who showed that the law was postponed when the input-output matrix was sparse.

The present model opens a new path to slowing down the law of large numbers. The sectoral business cycle model falls in with our framework of (S, s) models when coupled with sectoral lumpy adjustments in production. We show that the lumpy adjustments can generate sectoral propagation and thus significantly large aggregate fluctuations in production even when the sectoral shocks are independent. We also show that the aggregate fluctuations are exacerbated in a partial equilibrium setup, and hence we address the question of industrial dynamics when firms make discrete choices and interact with positive feedback.

The paper is organized as follows. Section 2 presents a simple model of the (S, s) economy. Section 3 shows the distribution of the aggregate. We present our sectoral business cycle model in Section 4 and show that the result in the (S, s) economy can be applied. Section 5 concludes. Proofs are deferred to the Appendix.

2 General Framework

An individual behavior is usually specified in such a way that the agent responds smoothly to a change in the environment. This behavioral specification may be expressed as follows:

$$x_i = Q_N(x) + e_i, \qquad i = 1, 2, \ldots, N. \tag{1}$$

Agent i's action is x_i and e_i is an agent specific factor. Define x and e as vectors with i-th coordinates x_i and e_i for $i = 1, 2, \ldots, N$ respectively. $Q_N(x)$ is an aggregator function of the action profile. We assume that $Q_N(x)$ is increasing in all the coordinates x_i, and $\partial Q_N(x)/\partial x_i = O(1/N)$. Thus an agent action generates a positive feedback effect of order $1/N$ on other agents' actions through Q_N. This behavioral function is commonly derived from a first-order condition of the agent's utility maximization. In a Cournot competition, for example, x_i is the best reply in producer i's production level given the other producer's production levels. In many economic decisions, however, the adjustment in individual behavior exhibits inertia and occasional lumpy correction:

$$\begin{cases} x_i = Q_N(x) + e_i - \lambda_i s_i \\ s_i \equiv ((Q_N(x) + e_i)(\text{mod } \lambda_i))/\lambda_i \end{cases}, \qquad i = 1, 2, \ldots, N \tag{2}$$

where $x(\text{mod } \lambda)$ denotes the remainder of the division of x by λ. By λ_i we denote the size of a lumpy adjustment in x_i. The variable s_i is normalized by the agent-specific bandwidth λ_i so that it always takes a value in $[0, 1)$. It represents the agent's position in the (S, s) band. The linear specification in e_i is standard as in Caballero and Engel (1991). The specification is versatile enough to allow, for example, the analysis of a standard sectoral business cycle model as we see in Section 4.

We construct a simple game to exemplify an economy that generates behavioral rules of the type (2). Consider a sequence of strategic games $G_N = \langle N, (X_i), (U_i^N)\rangle$, $N = N_0, N_0 + 1, \ldots$ for a large integer N_0. A game G_N is played by N players. The action set for player i is $X_i = \{0, \pm\lambda_i, \pm 2\lambda_i, \ldots\}$. Player i's payoff is given by a function $U_i^N(x) = -(Q_N(x) + e_i - \lambda_i/2 - x_i)^2$ which is a quadratic loss function utilized in Caplin and Leahy (1997). The payoff attains its maximum, zero, when $x_i = Q_N(x) + e_i - \lambda_i/2$. The optimal $x_i \in X_i$ lies between $\chi - \lambda_i$ and χ, where χ satisfies $U_i^N(\chi - \lambda_i, x_{-i}) = U_i^N(\chi, x_{-i})$, by the concavity of U_i^N and by $\partial Q_N(x)/\partial x_i = O(1/N)$. Hence $x_i = \chi - \chi(\text{mod } \lambda_i)$ is the global maximum for large N. Solving χ, we obtain (2) as the best response of i. Thus a solution x of the system (2) is a Nash equilibrium of the game G_N. When $\lambda_i = 0$, we reset the action set as an entire real line: $X_i = R$. Then the first-order condition of i's payoff maximization produces (1) as a behavioral rule for a smoothly adjusting case.

The existence of a solution x for the system of behavioral rules (2) is readily available. Let an underline and an overline denote a lower and an upper bound of the variable, respectively.

Lemma 1 (Existence of equilibrium) *Suppose that $Q_N(x)$ is increasing in $x \in R^N$ and that e_i and λ_i are bounded. Suppose that there exist scalars $\underline{x}$ and $\overline{x}$ that satisfy $\underline{x} = Q_N(\underline{x}, \underline{x}, \ldots, \underline{x}) + \underline{e} - \overline{\lambda}$ and $\overline{x} = Q_N(\overline{x}, \overline{x}, \ldots, \overline{x}) + \overline{e}$. Then the system (2) has a solution for any e and λ.*

Proof: If $Q_N(x)$ is increasing in x, then $Q_N(x)+e_i-(Q_N(x)+e_i)(\text{mod } \lambda_i)$ is increasing in x as well. Let us construct a vector function by stacking N functions in (2). Since $0 \leq (Q_N(x) + e_i)(\text{mod } \lambda_i) < \lambda_i$, the constructed $\underline{x}$ and $\overline{x}$ defines the lower and upper bound of x_i. The vector function thus has a fixed point by Tarski's theorem (Vives (1990)). Any $x_i \in R$ that satisfies (2) belongs to X_i by construction. □

Lemma 1 implies that the equilibrium exists when Q_N is increasing and when the smooth counterpart (1) of (2) has an equilibrium for an extended space $[\underline{e} - \overline{\lambda}, \overline{e}]$ of the exogenous variable e_i. Since $Q_N(x)$ is increasing, players are situated in strategic complementarity. Thus, G_N is a particular case of supermodular games.

Suppose that x^1 and x^0 correspond to the equilibria that solve the system (2) given e^1 and e^0, respectively. We generate e^1 by a perturbation ϵ_i/N of e^0, as $e_i^1 = e_i^0 + \epsilon_i/N$ where ϵ_i is positive and i.i.d. across i. The perturbation shock is normalized by N so that its impact matches the magnitude of the feedback effect $\partial Q_N(x)/\partial x_i$. Define a propagation size caused by the perturbation as $Q_N(x^1) - Q_N(x^0)$. This is the increment in the aggregate index Q_N. It is also the common factor of increments in individual actions x_i. Our goal is to asymptotically characterize the distribution of the propagation size for large N.

The system (2) allows multiple equilibria. To complete the definition of the perturbation, we need an equilibrium selection algorithm. We employ best response dynamics as such algorithm. Define u as the step of the dynamics $(x_{i,u}, s_{i,u})$. Set the initial value of the best response dynamics equal to the initial equilibrium value: $x_{i,0} = x_i^0$ and $s_{i,0} = s_i^0$. For subsequent steps, we define the individual responses as follows:

$$x_{i,1} = \begin{cases} x_{i,0} + \lambda_i & \text{if } s_{i,0} + \epsilon_i/(N\lambda_i) \geq 1 \\ x_{i,0} & \text{otherwise} \end{cases} \tag{3}$$

$$x_{i,u+1} = \begin{cases} x_{i,u} + \lambda_i & \text{if } s_{i,u} + (Q_N(x_u) - Q_N(x_{u-1}))/\lambda_i \geq 1 \\ x_{i,u} & \text{otherwise} \end{cases}, \quad \text{for } u \geq 1 \tag{4}$$

$$s_{i,1} = s_{i,0} + (\epsilon_i/N - x_{i,1} + x_{i,0})/\lambda_i \tag{5}$$

$$s_{i,u+1} = s_{i,u} + (Q_N(x_u) - Q_N(x_{u-1}) - x_{i,u+1} + x_{i,u})/\lambda_i, \quad \text{for } u \geq 1. \tag{6}$$

Define T as a stopping time of the process $Q_N(x_u) - Q_N(x_{u-1})$. Namely, $T \equiv \min_{\{u|Q_N(x_u)-Q_N(x_{u-1})=0\}} u$. Then x_T satisfies the system (2) for e^1. Thus the best response dynamics constitutes an equilibrium selection algorithm that defines an equilibrium $x^1 = x_T$. The stopping time T is finite with probability one when $N \to \infty$, as we will prove when we show the propagation distribution. The best response dynamics reflects a one-sided (S, s) policy.[2] This equilibrium selection algorithm has been used by Vives (1990) and Cooper (1994). The algorithm has a straightforward economic intuition. We start from an initial equilibrium that solves the system of best response functions, and add a disturbance to the system. Then we update the individual choice by applying the best response function iteratively until a new solution is reached. This particular equilibrium selection has some immediate implications. The procedure imposes inertia on the individual actions. Agents do not adjust their actions unless they are strictly better off by adjusting. Also, it rules out an equilibrium far from an initial equilibrium that would require some kind of informational coordination. The preclusion of big jumps in equilibrium based on informational coordination suits our aim to focus on strategic complementarity alone as a propagation mechanism.

3 Main Results

In this section we derive the distribution of propagation size under a simplifying assumption (Assumption 2). Let us define a perturbation experiment.

Assumption 1 (Perturbation) *We have an equilibrium x^0. In a perturbation $e_i^1 = e_i^0 + \epsilon_i/N$, ϵ_i is positive and bounded. An agent's position s_i^0 is a random variable with support $[0, 1)$ and i.i.d. across i. The cumulative distribution function of s_i^0 satisfies $\lim_{h\to 0}(F_s(1) - F_s(1-h))/h = \psi < \infty$.*

By this assumption, the perturbation shock ϵ_i/N is positive and thus the agents follow a one-sided (S, s) policy. We let s_i^0 follow any distribution that has a density in the vicinity of the border $s_i^0 = 1$. Our initial condition of the perturbation is an equilibrium x^0 and random variables e_i^0, such that $s_i^0 = (e_i^0 - x_i + Q_N(x))/\lambda_i$ obeys the distribution function F_s where λ_i, $i = 1, 2, \ldots, N$, is a prefixed sequence.[3] This formalization

[2] A two-sided policy can be similarly analyzed in our framework.

[3] The assumption that s_i is identically distributed holds for any distribution of e_i when the equilibrium is symmetric: $x_i = x$ and $\lambda_i = \lambda$ for all i. The assumption imposes a restriction on e_i when

represents the situation of an econometrician who observes the equilibrium action x^0 but does not observe the environment e^0 or the position variable s^0. The econometrician only knows the likelihood of e^0 among all vectors e^0 that could support x^0.

As a simplifying assumption, we assume that the aggregator $Q_N(x)$ is symmetric and linear in x_i, and the lumpiness of adjustments is the same for all agents.

Assumption 2 (Linearity and homogeneity) *$Q_N(x) = \phi \sum_{i=1}^N x_i/N$ where $\phi\psi \leq 1$. Also $\lambda_i = \lambda$ for all i.*

The existence of solution x^0 is confirmed by Lemma 1 under Assumption 2 when $\phi < 1$, for $[(\underline{e} - \lambda)/(1-\phi), \overline{e}/(1-\phi)]^N$ is the compact domain of x that is mapped into itself.

Define $\mu = \mathrm{E}[\epsilon_i]\psi/\lambda$. We then obtain the distribution of the propagation size.

Proposition 1 (Propagation distribution) *Under Assumptions 1 and 2, the normalized propagation size $N(Q_N(x^1) - Q_N(x^0))$ converges in distribution to $w\phi\lambda$ when $N \to \infty$, where w is a random variable that follows a probability distribution:*

$$\Pr(w) = (\phi\psi w + \mu)^{w-1} \mu e^{-\phi\psi w - \mu}/w! \tag{7}$$

for $w = 0, 1, 2, \ldots$. The moment generating function of w is $e^{\mu(G(s)-1)}$ where $G(s)$ is a moment generating function which satisfies a functional equation $G(s) = e^{s+\phi\psi(G(s)-1)}$. The distribution of w is infinitely divisible. Its tail is approximated by:

$$\Pr(w) \approx (\mu e^{-\mu}/(\phi\psi\sqrt{2\pi}))(\phi\psi e^{1-\phi\psi})^w w^{-1.5} \tag{8}$$

for large integer w.

Proof: See Appendix A.1.

The propagation size $Q_N(x^1) - Q_N(x^0)$ has an asymptotic mean $(\mathrm{E}[\epsilon_i]/N)\phi\psi/(1-\phi\psi)$ and variance $(\mathrm{E}[\epsilon_i]/N^2)\psi\lambda\phi^2/(1-\phi\psi)^3$. The distribution starts out as a power-law distribution $w^{-1.5}$ at $w = 0$ and exhibits an exponential truncation $(\phi\psi e^{1-\phi\psi})^w$ at the tail. The distribution becomes a pure power-law when $\phi\psi \to 1$ and its variance tends to infinity.

Proposition 1 marks a clear departure from an economy without inertia. Let us set up a model of a smoothly adjusting counterpart of our economy. Suppose that an agent's optimal decision follows a smooth adjustment rule (1). Then, under Assumption

the equilibrium is asymmetric. A particular distribution of e_i that satisfies the assumption is a uniform distribution over $[0, \lambda_i z_i)$ where z_i is an arbitrary integer. Lemmas 2 and 3 offer an alternative justification that the assumption holds at the stationary state when e_i follows a random walk.

2, we directly obtain that $Q_N(x^1) - Q_N(x^0) = (\phi/(1-\phi))\sum_i \epsilon_i/N^2$ when $\phi < 1$. By the central limit theorem, a normalized propagation size $N^{1.5}(Q_N(x^1) - Q_N(x^0) - (\mathrm{E}[\epsilon_i]/N)\phi/(1-\phi))$ asymptotically follows a normal distribution with mean zero and variance $\mathrm{Var}[\epsilon_i]\phi^2/(1-\phi)^2$. The normalization factor $N^{1.5}$ consists of two factors: the contribution from the normalized shock (N) and the central limit theorem ($N^{0.5}$). The propagation size in Proposition 1 has the normalization factor N, that is equal to the normalization factor for the smooth case less the contribution of the central limit theorem. An immediate implication of the comparison between the two economies is that the aggregate in the lumpy economy is much more volatile than its smooth counterpart. The asymptotic variance of the propagation size $Q_N(x^1)-Q_N(x^0)$ is scaled to N^{-3} in the smooth economy, whereas it is scaled to N^{-2} in the lumpy economy. Thus, the variance of the lumpy economy converges to zero N-times slower than its smooth counterpart. This leads to the possibility that the threshold model may quantitatively explain the large magnitude of aggregate fluctuations when a smooth model cannot.

Besides the convergence rate of the variance, the distribution also differs in its shape. It is skewed and heavy-tailed in the lumpy economy, whereas it follows a normal distribution in the smooth economy. The heavy-tail accounts for an extra $1 - \phi\psi$ in the denominator in the variance. Also, the aggregate variance is not affected by the variance of perturbation shocks in the lumpy economy, whereas it is directly affected in the smooth economy. In fact, the aggregate variance in the lumpy economy remains unchanged when the shock ϵ_i is replaced with its mean. The randomness of the shock is irrelevant in determining the fluctuation magnitude. To the contrary, the distribution of s_i^0 directly affects the variance through ψ. This implies that the configuration of agents' positions in the inaction intervals is the crucial variable in determining the variability of propagation sizes. The asymptotic mean of the propagation size is the same for both economies when $\psi = 1$. The difference between the two economies thus lies in the second and higher moments.

It is useful to compare our finite agent model with continuum of agents models. Let us set up the continuum of agents model as follows. Suppose that $Q(x) = \phi \int x_i di$. Consider a perturbation ϵ_i/N on e_i for some fixed N. Under Assumption 2, the best response is $x_i^1 = x_i^0 + \lambda$ if $s_i + (\epsilon_i/N + \phi \int x_i^1 - x_i^0 di)/\lambda \geq 1$ and $x_i^1 = x_i^0$ otherwise, if N is large enough that no agent adjusts more than λ. Suppose that s_i^0 is uniformly distributed. Then the fraction of agents that adjust is a deterministic value $(\mathrm{E}[\epsilon_i]/N + \phi \int x_i^1 - x_i^0 di)/\lambda$. Thus, we obtain that $Q(x^1) - Q(x^0) = (\mathrm{E}[\epsilon_i]/N)\phi/(1-\phi)$. This is a deterministic value that is equal to the mean of the propagation size in our finite model. The continuum of agents model does capture the mean impact of the propagation, but fails to capture its stochastic nature. The propagation size in the continuum of agents model is also equal to the mean propagation size in the smooth economy, thus we

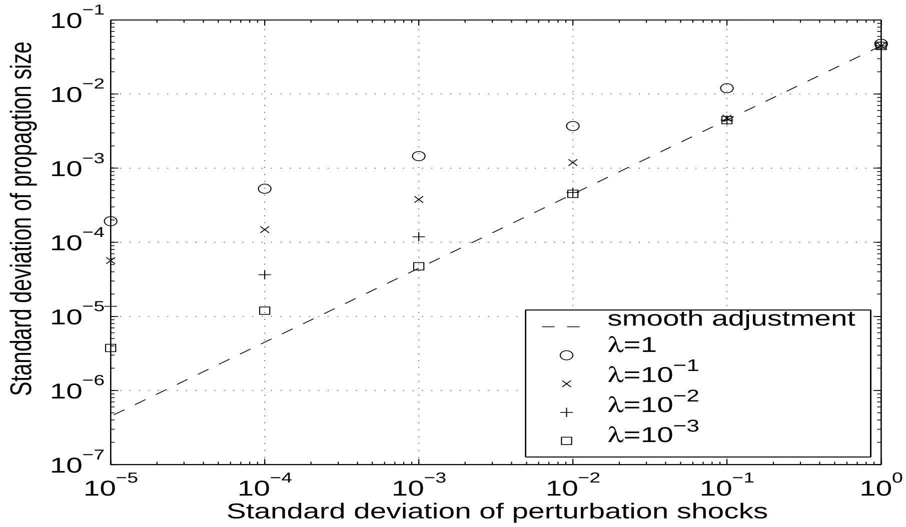

Figure 1: Simulation for cross-over points

have reproduced the neutrality theorem. Note that N is fixed in the calculation above whereas the number of agents is uncountably infinite. This is the situation where a feedback effect of an agent's action on the aggregate is overwhelmed by the exogenous shock, and is analogous to the case when the shock dominates the lumpiness in our finite economy.

Above comparison with a smooth economy or a continuum of agents economy suggests that our fluctuation result depends on the size of the lumpiness λ relative to the size of the shock ϵ_i/N. Intuitively speaking, the lumpiness is washed away when the shock overwhelms the lumpiness. A simulation verifies this intuition. Figure 1 plots the standard deviation of the propagation size for various sizes of λ and $\sigma_e \equiv \mathrm{Std}[\epsilon_i/N]$.[4] It is seen that the standard deviation of the propagation increases proportionally with σ_e when $\sigma_e > \lambda$ so that it behaves as in the smooth economy, whereas it significantly deviates upward when $\sigma_e < \lambda$.

[4]The simulation is executed as follows. Draw an initial value of e_i^0 for $i = 1, 2, \ldots, N$ from a distribution uniform over $[0, \lambda)$. Then $x_i^0 = 0$ for all i is an equilibrium. Draw a perturbation ϵ_i from a normal distribution with mean zero and various standard deviations σ_e. The new equilibrium x^1 is calculated by the best response dynamics (3,4). Then we obtain a propagation size $\phi \sum_{i=1}^{N}(x_i^1 - x_i^0)/N$. We set $N = 500$ and $\phi = 0.5$. We compute the propagation size for 10^4 times and the standard deviation of the propagation size for each value of λ.

We can deduce a time-series property of the aggregate fluctuations. The infinite divisibility of distribution (7) plays an important role here. In general, a random variable having an infinitely divisible distribution is equivalent to the random variable being an increment of a stochastic process with independent increments (Feller, 1966, page 177). We can see this point for our case. Suppose that e_i^t evolves as a stochastic process with independent and positive increments. We define an equilibrium path as a sequence of static equilibria. Then we can define the aggregate growth, $Q_N(x^t) - Q_N(x^0)$, for every time horizon t. The variance of a cumulative innovation $e_i^t - e_i^0$ grows linearly with the time horizon t. For a fixed large N, suppose that ϵ_i/N is equivalent to $e_i^1 - e_i^0$ which is an increment of e_i for a unit time horizon. Then, the normalized growth $N(Q_N(x^1) - Q_N(x^0))$ follows the moment generating function $e^{\mu(G(s)-1)}$, which is a compound Poisson distribution with Poisson mean μ and a random variable that follows $G(s)$. Now consider a growth in a shorter time horizon $N(Q_N(x^{1/n}) - Q_N(x^0))$ for any integer n. Then this growth follows a moment generating function $e^{(\mu/n)(G(s)-1)}$, which is a compound Poisson with the same random variable following $G(s)$ and with a Poisson mean μ/n which is linearly scaled by the time horizon $1/n$. Therefore, the sequence of the static equilibria of an economy with a fixed N is approximated by a compound Poisson process with hazard rate μ and a random variable that follows $G(s)$ for a time horizon shorter than the unit time. By a similar argument, a smoothly adjusting counterpart is approximated by a Brownian motion.

The compound Poisson process obtains only for a vanishingly short time scale. This fact corresponds to that Proposition 1 holds only when the shock is small relative to the lumpiness. Our simulation in Figure 1 shows, however, that the deviation from the smooth economy is observable for $\sigma_e < \lambda$. This implies in the time-series context that the aggregate process is in transition from a compound Poisson to a Brownian up to the time scale for which the size of the accumulated shock matches the lumpiness. The cross-over time scale is characterized by λ/σ_e where σ_e is the size of shocks for a unit time.

Proposition 1 holds under various distributions of s_i^0. It is necessary, however, to know the stationary distribution of s_i, when we consider a sequence of static equilibria. We can indeed show that the uniform distribution is a stationary distribution of s_i with respect to the perturbation. The following Lemma is a reexpression of the result of Caplin and Spulber (1987).

Lemma 2 (Stationary distribution of s_i) *Consider a perturbation $e_i^1 = e_i^0 + \epsilon_i$ where ϵ_i is bounded. Suppose that s_i^0 is independently and uniformly distributed over $[0, 1)$. Assume that $Q_N(x^1) - Q_N(x^0)$ is asymptotically independent of s_i^0 when $N \to \infty$. Then, s_i^1 is independently and uniformly distributed asymptotically when $N \to \infty$.*

Proof: See Appendix A.2.

Lemma 2 shows that s_i stays uniformly distributed even after a sizable shock ϵ_i hits. The assumption of the asymptotic independence between $Q_N(x^1) - Q_N(x^0)$ and s_i^0 is natural, for we consider the situation in which the effect of a single agent's action is of order $O(1/N)$ in aggregation.

The uniform distribution is also a convergent point of s_i if e_i evolves as a random walk. The following Lemma is analogous to Caballero and Engel (1991).

Lemma 3 (Convergence of s_i) *Consider that e_i evolves as a random walk $e_{i,t+1} = e_{i,t} + \epsilon_{i,t}/N$ for $t = 0, 1, \ldots$ where $\epsilon_{i,t}$ is i.i.d. across both i and t. Pick $p > 2$ arbitrarily. Assume that $Q_N(x_{t=N^p}) - Q_N(x_0)$ and $\sum_{t=0}^{N^p-1} \epsilon_t/(N\lambda_i)$ are asymptotically independent. Then $s_{i,t=N^p}$ converges in distribution to a distribution uniform over a unit interval and independent across i when $N \to \infty$.*

Proof: See Appendix A.3.

Above Lemmas imply that the uniform distribution of s_i serves as a steady state of the economy with respect to its aggregate variability when the environment e_i is diverging. Thus $\psi = 1$ is the stationary value of ψ. The condition $p > 2$ in Lemma 3 implies that the convergence takes no less than N^2 periods for which the accumulated shock achieves an observable size of standard deviation $\mathrm{Std}[\epsilon_i]/\lambda_i$.

The Lemmas are proved for nonlinear $Q_N(x)$ and heterogeneous λ_i. In fact, Proposition 1 can also be shown in the general environment. We discuss this generalization later in the next section.

4 An Application

In this section, we apply our general result to a sectoral business cycle model in a general equilibrium setting. Our sectoral model is standard as seen in Galí (1994) and Horvath (2000), except that we impose a lumpy adjustment of an input of production. We analyze a static model for the sake of simplicity. We employ one approximation for aggregate consumption to obtain a closed-form solution of an equilibrium. We relate the economic fundamental parameters to the parameters of propagation distribution. We also discuss an implication under a partial equilibrium setting and its connection with so-called critical phenomena.

The economy consists of a representative household and N monopolistic firms. Firm i produces good i, $i = 1, 2, \ldots, N$. The representative household solves the following maximization problem:

$$\max_{c_i,L} \frac{C^{1-\rho}}{1-\rho} - \frac{L^{1+\nu}}{1+\nu} \tag{9}$$

subject to

$$\sum_{i=1}^{N} p_i c_i = L + \sum_{i=1}^{N} \pi_i \tag{10}$$

$$C = N^{1/(1-\sigma)}(\sum_{i=1}^{N} c_i^{(\sigma-1)/\sigma})^{\sigma/(\sigma-1)} \tag{11}$$

where $\sigma > 1$, $\rho \geq 0$, and $\nu \geq 0$. We take leisure as a numeraire. It is assumed that the time constraint for labor supply is not binding at equilibrium. C is a composite good for consumption composed of all N goods. For simplicity, all goods are symmetrically substitutable with a constant elasticity of substitution σ, as in Kiyotaki (1988) and Galí (1994).

Monopolistic firm i owns production technology:

$$y_i = A_i(f_i^{1-\alpha} Z_i^{\alpha})^{\theta} L_i^{\gamma}. \tag{12}$$

We allow the production function to be either increasing or decreasing returns to scale. To ensure the existence of a solution to the monopolist's problem, we set the upper bound of the returns as $\theta + \gamma < \sigma/(\sigma - 1)$. L_i is a labor input. Z_i is a composite intermediate input that combines all N goods exactly as in C. We assume that the elasticity of substitution is the same for the consumer's demand and the producer's factor demand to render the model solvable. Let $z_{i,j}$ denote the component good j which is used by Z_i. Then,

$$Z_i = N^{1/(1-\sigma)}(\sum_{j=1}^{N} z_{i,j}^{(\sigma-1)/\sigma})^{\sigma/(\sigma-1)}. \tag{13}$$

Input f_i is also a composite input that is produced by an identical technology to that of Z_i up to a sector-specific coefficient B_i:

$$f_i = B_i N^{1/(1-\sigma)}(\sum_{j=1}^{N} (z_{i,j}^{f})^{(\sigma-1)/\sigma})^{\sigma/(\sigma-1)}. \tag{14}$$

Unlike Z_i, however, f_i can only take discrete values:

$$f_i \in \{1, e^{\pm\lambda_i}, e^{\pm 2\lambda_i}, \dots\} \tag{15}$$

for some fixed $0 < \lambda_i < \log 2$. Thus f_i is an indivisible input. Typically, big equipment or plants are such inputs. Henceforth we call f_i plants. Along with this interpretation, the production function can be rewritten in terms of a plant:

$$y_i = f_i^{\theta+\gamma} A_i (Z_i/f_i)^{\alpha\theta} (L_i/f_i)^{\gamma}. \tag{16}$$

We regard $A_i(Z_i/f_i)^{\alpha\theta}(L_i/f_i)^{\gamma}$ as a plant-level technology that allows smooth adjustment of inputs Z_i and L_i. We assume $\alpha\theta + \gamma < 1$ so that the plant-level technology exhibits decreasing returns to scale. The f_i expresses replication of the plant technology. Due to the discreteness of f_i, the production function exhibits periodic local convexity. Overall returns to scale is given by $\theta + \gamma$. This discrete adjustment in sector-level production is the key assumption in this sectoral model.

The monopolistic firm's problem is defined as follows:

$$\pi_i \equiv \max_{f_i, L_i, z_{i,j}, z_{i,j}^f, p_i, y_i} p_i y_i - L_i - \sum_{j=1}^{N} p_j z_{i,j} - \sum_{j=1}^{N} p_j z_{i,j}^f \tag{17}$$

constrained by a demand function for good i and the production functions (12, 13, 14, 15).

An equilibrium of the economy, given the exogenous variables A_i, B_i, λ_i, is a price vector p_i and an allocation $(c_i, y_i, L_i, z_{i,j}, z_{i,j}^f)$ that satisfies the representative household's maximization problem (9), the monopolists' problems (17), and market clearing conditions for goods and labor:

$$y_i = c_i + \sum_{j=1}^{N} z_{j,i} + \sum_{j=1}^{N} z_{j,i}^f, \quad \forall i \tag{18}$$

$$L = \sum_{j=1}^{N} L_j \tag{19}$$

Derivation and characterization of the equilibrium is deferred to Appendix A.4. Here we focus on a reduced form representation of the system. An inaction region defined by (39) in Appendix A.4 determines f_i uniquely given other firms' decision. It is explicitly solved as:

$$f_i = e^{\lambda_i \lfloor (D_i + \xi_a \log A_i + \xi_b \log B_i + \phi \log S(A,f))/\lambda_i \rfloor} \tag{20}$$

where $\lfloor x \rfloor$ is an operator that takes the largest integer less than x. S is defined as:

$$S(A,f) \equiv \left(\sum_k (A_k f_k^{(1-\alpha)\theta})^{(\sigma-1)/\xi_1} / N\right)^{\xi_1/((\sigma-1)(1-\alpha)\theta)}. \tag{21}$$

We will often suppress the argument A in $S(A, f)$. $D_i, \xi_a, \xi_b, \xi_1, \phi$ are constants defined in Appendix A.4. Let us also define a remainder of the operator in (20) as:

$$s_i = ((D_i + \xi_a \log A_i + \xi_b \log B_i + \phi \log S(f))/\lambda_i)(\text{mod } 1). \tag{22}$$

By definition, $0 \leq s_i < 1$. This s_i represents the distance between the equilibrium number of plants f_i and the point where the number of plants would be decreased to $f_i e^{-\lambda_i}$.

Equations (20,22) fall in with our general framework (2). The logarithm of the plant size $\log f_i$ is the control variable x_i, the idiosyncratic productivity factor $D_i + \xi_a \log A_i + \xi_b \log B_i$ corresponds to e_i, and $\phi \log S(f)$ is the aggregator $Q(x)$. Parameter ϕ represents a degree of strategic complementarity among firms' decisions on plants. The bigger ϕ is, the more impact $S(f)$ has on an individual f_i. This is a pecuniary externality where an increase in a plant level in an industry leads to a cheaper price on the good and then induces larger supply of other industries that use the good as an intermediate input. Before we proceed to apply our result to the sectoral model, let us examine the existence of an equilibrium given the exogenous variables (A_i, B_i, λ_i) for the case $\phi < 1$. Construct $\overline{f} = (e^{\overline{D}} \overline{A}^{\xi_a + \phi/((1-\alpha)\theta)} \overline{B}^{\xi_b})^{1/(1-\phi)}$ and $\underline{f} = (e^{\underline{D}} \underline{A}^{\xi_a + \phi/((1-\alpha)\theta)} \underline{B}^{\xi_b} / e^{\overline{\lambda}})^{1/(1-\phi)}$. Stack N equations of the best reply (20) into a new function. Then the function maps $[\underline{f}, \overline{f}]^N$ into itself, and Lemma 1 applies.

Define an output composite good as:

$$Y = N^{1/(1-\sigma)} (\sum_{k=1}^{N} y_k^{(\sigma-1)/\sigma})^{\sigma/(\sigma-1)}. \tag{23}$$

Suppose that we have an initial equilibrium denoted by a superscript 0. The initial aggregate production is Y^0. A perturbation is performed by adding a technological shock ϵ_i/N to $\log A_i$ for all i. A new equilibrium is defined by the equilibrium selection algorithm. Let Y^1 denote the new aggregate production. Define $g_N = \log Y^1 - \log Y^0$. Thus g_N is the growth rate of the aggregate production Y induced by a perturbation ϵ_i/N for an economy with N firms.

Now we show that the propagation result in our general framework holds in the sectoral model. First we derive an explicit distribution function of the growth rates for a special case where the elasticity of substitution σ goes to one and the lumpiness λ_i is common across firms. Composite functions of C, Z, f is Cobb-Douglas at the limit $\sigma = 1$, although the monopolistic pricing diverges at the limit. We perturb the exogenous variables of the cost of plants B_i. More natural case where the total factor productivity A_i is perturbed is studied later in a generalized proposition.[5]

[5]The reason we perturb B_i here is that $\xi_a \to 0$ holds when $\sigma \to 1$. This means that the plant

Define $\mu = \mathrm{E}[\epsilon_i]\psi/\lambda$ for a constant λ. Define a constant $c_0 = (1-\alpha)\theta/((1-\alpha\theta-\gamma(1-\rho)/(1+\nu))\phi)$. Then we obtain the following:

Proposition 2 (Production growth distribution) *Assume that $\lambda_i = \lambda$ for all i. The sequence A_i, $i = 1, 2, \ldots$, is bounded. Suppose that we have an equilibrium f^0. B_i^0 is distributed so that s_i^0 has a cumulative distribution function F_s that satisfies $\lim_{h\to 0}(F_s(1) - F_s(1-h))/h = \psi$. Consider a perturbation $\log B_i^1 = \log B_i^0 + \epsilon_i/N$ where ϵ_i is i.i.d. across i, positive, and bounded. Suppose that we take a limit $N \to \infty$ first and then $\sigma \to 1$. Then, for $\phi\psi \leq 1$, the scaled growth rate of aggregate production $Ng_N/(c_0\lambda\phi)$ asymptotically follows a distribution function (7) in Proposition 1.*

Proof: See Appendix A.5.

This result provides us with a foundation of a technology-driven business cycle theory. It has been thought that finely disaggregated idiosyncratic shocks on technology do not add up to a sizable aggregate fluctuation that can match the actual magnitude of business cycles. When the sectoral production exhibits lumpy adjustments, we show that sector-specific technological shocks are amplified by a propagation of adjustments and generate significant aggregate fluctuations. This point becomes clearer when we compare the variance of the lumpy economy with its smooth counterpart. Suppose that f_i can take any positive real number instead of the discrete values as in (15). Then the optimality of f_i is characterized by a usual first-order condition instead of an inaction region (35). Define an equilibrium and a perturbation experiment of the smooth economy accordingly. Then, in the smoothly-adjusting counterpart of our economy, a normalized growth rate $N^{1.5}g_N$ asymptotically follows a non-degenerate normal distribution. The asymptotic variance of growth rates in the smooth economy is of order N^{-3} instead of N^{-2} in the lumpy economy. Hence the growth rate converges to a deterministic value N-times faster than the case of the lumpy adjusting economy.

The propagation distribution is infinitely divisible, which allows us to regard the propagation as an independent increment of the evolution of aggregate production. The propagation distribution is skewed and heavy-tailed when the shock is smaller than the size of lumpiness, whereas it follows a normal distribution when the shock overwhelms the lumpiness. Taking into account that the process of sectoral technology naturally follows a process with independent increments, the transition of the phases can be viewed in terms of time horizons. The aggregate growth rate is more volatile than its smooth counterpart for a short time horizon, whereas it is just as volatile as

level f_i will be insensitive to A_i at the limit. As in (39), both $\log A_i$ and $\log B_i$ linearly determine the inaction region of f_i. Thus a perturbation of $\log B_i$ can be symmetrically understood as that of $\log A_i$.

the smooth counterpart for a long time horizon. Also, as in Lemma 2, the stationary distribution of s_i is uniform. The simulation shows that, at the stationary distribution, the transition occurs at the time horizon for which the lumpiness size is equal to the standard deviation of the technology growth. These facts point to that the lumpy model is relevant to the short-run fluctuations.

Further investigation on the time-series property, especially the autocorrelation structure, would require a dynamic extension of the sectoral model. The extension is important, since the investment exhibits lumpy adjustments most notably and is often considered to be a driving force of business cycles. The extension is also straightforward. We replace the discrete input f_i with capital and impose a discreteness condition on the investment rate. Then, the formula for the inaction region (35) has an additional term associated with an interest rate. The interest rate behaves as another dampening factor on the propagation along with the real wage, which reflects intertemporal consumption smoothing. The dampening effect cancels out over time, however, since the interest rate comes back to its steady state level where the capital ceases to accumulate. Thus, the difference of steady-state production levels caused by a perturbation follows the distribution of the propagation we have shown. This detail is found in a separate paper Nirei (2002).

Besides recapturing the slow convergence of fluctuations with respect to N, the distribution formula (7) exhibits a non-normal, heavy tailed distribution that converges to a power-law distribution as $\phi \to 1$. This property is well understood under the light of so-called critical phenomena. The propagation process $S_N(x_u) - S_N(x_{u-1})$ is actually a branching process with mean ϕ as we see in the proof. Thus the behavior of the propagation process under the stationary distribution of s_i^0 (and thus $\psi = 1$) is essentially identical to a percolation on a Bethe lattice (Grimmett, 1999, page 254) with $1 - \phi$ being the difference of the probability to a critical probability. As in Grimmett (1999), the cluster volume in the percolation follows a power law with exponent -1.5 at the criticality $\phi = 1$, which corresponds to our power law in (8) for $\phi = 1$. Also, the second moment of the cluster is proportional to an inverse cube of the difference to the critical probability, just as we have $(1 - \phi)^{-3}$ in the asymptotic variance of the growth rate.

The criticality condition, $\phi = 1$, is equivalent to a particular alignment of model parameters $\theta + \gamma - 1 = \tilde{\rho}\gamma$ where $\tilde{\rho} \equiv (\rho + \nu)/(1 + \nu)$. Such an alignment includes some important cases. One case is that $\tilde{\rho} = \theta = 1$ with any value for the labor share γ. The returns of commodity inputs Z and f is constant when $\theta = 1$. Here $\tilde{\rho}$ represents the elasticity of the equilibrium total product Y with respect to the equilibrium real wage $1/P$. Also $\tilde{\rho} = 1$ occurs when $\rho = 1$ with any ν or when $\nu \to \infty$ with any ρ. Thus it occurs either when the utility of the commodity consumption is logarithmic or when the labor supply is inelastic. Another interesting case is the case when the utility of consumption and leisure are linear ($\rho = \nu = 0$). In this case, the criticality obtains when the overall returns to scale is constant ($\theta + \gamma = 1$). These relations may be summarized as a balancing of two contradicting forces. The increasing returns in the production side accelerates the fluctuations, whereas the sensitivity of the labor supply to real wage dampens them. The criticality occurs when the acceleration force is just as strong as the dampening one. We can also interpret the criticality in terms of ϕ which is a behavioral parameter. As we discussed before, ϕ represents the degree of strategic complementarity among firms' choices of plants. At $\phi = 1$, the strategic complementarity is perfect when a response function is viewed in a global range. We mean by perfect complementarity that a proportional increase in f for all the other sectors induces the same proportional increase in a sector, if the increment is larger than the inaction interval. On the other hand, the response function exhibits no complementarity in a local range. A shock smaller than the width of the inaction region does not cause a symmetric movement across firms because of the lumpy behavior. The power-law distribution of aggregate growth rates is induced by the perfect complementarity in a global range and no complementarity in a local range.

The possibility of a power-law distribution of sectoral propagation was first pointed out by Bak, Chen, Scheinkman, and Woodford (1993) along the line of literature of self-organized criticality. The point of the literature is that the critical phenomena, which are broadly associated with power-law distributions, can occur at the sink of a class of dynamical systems, whereas such criticality had been believed to require a fine tuning of parameters. The "self-organization" mechanism to arrive at a critical point is expressed in our model as a convergence of s_i to a uniform distribution. The critical distribution, however, requires $\phi = 1$ as well as $\psi = 1$. The dampening force $\phi < 1$ that exponentially truncates the power-law propagation in our model stems from decreasing returns of labor which Bak, Chen, Scheinkman, and Woodford (1993) abstracted. This point can be seen by our formula of criticality $\theta + \gamma - 1 = \tilde{\rho}\gamma$. As a labor share ($\gamma$) becomes negligibly small, the required returns to scale ($\theta + \gamma$) decreases to one. The production function can be interpreted as a plant-level technology, whose returns to scale is $\alpha\theta + \gamma$, replicated by the number of plants, whose returns to scale is $\theta + \gamma$. Thus, when $\gamma \to 0$ and $\theta \to 1$, our model represents the economy where a decreasing-returns-to-scale plant technology can be replicated at a constant cost of a plant. Labor is the only input that is not produced in our model economy. Thus the wage generally rises as the aggregate production increases. In this general equilibrium setup, the critical propagation occurs only when some increasing returns to scale offsets the dampening effects of a rising wage.

The critical phenomena emerge more naturally in a partial equilibrium setup for the commodity market. Let us imagine that real wage is given or exhibits rigidity. The real rigidity can be seen as a case where $\rho = \nu = 0$, namely the utility function is linear. When the real wage $(1/P)$ is fixed, the criticality holds for the case of constant returns to scale, $\theta + \gamma = 1$. This is shown by the inaction region (35), where the degree of strategic complementarity with fixed wage is expressed by ξ_s which is equal to one when $\theta + \gamma = 1$. Hence, the argument of self-organized criticality in sectoral production linkage is valid when the product markets and the labor market are relatively independent so that the resource constraint of labor does not cause a dampening effect on the pecuniary externality among production sectors. Thus, in the partial equilibrium setting, a group of industries can experience critical fluctuations even in the case of constant returns to scale, when the system is characterized by threshold behavior and strategic complementarity across industrial units.

We can show that our propagation result obtains under a relaxed assumption and with a perturbation on the total factor productivity A_i. Define the following functions:

$$a_i(f) \equiv \frac{\phi(\hat{A}_i^1 f_i^{(1-\alpha)\theta})^{(\sigma-1)/\xi_1}}{\lim_{N\to\infty}\sum_{j=1}^N (A_j^1 f_j^{(1-\alpha)\theta})^{(\sigma-1)/\xi_1}/N} \tag{24}$$

$$b(\lambda(k)) \equiv e^{\lambda(k)} - 1 + \sum_{n=2}^{\infty}(e^{\lambda(k)}-1)^n (c_1-1)(c_1-2)\cdots(c_1-n+1)/n! \tag{25}$$

where $\hat{A}_i^1$ is a deterministic value of A_i that solves (20) and (22) for f^0 and $s_i = 1$, and c_1 is a constant $c_1 \equiv (1-\alpha)\theta(\sigma-1)/\xi_1$. Let us redefine J_{x^0} as a moment generating function of $a_i(f^0)$ when i is randomly chosen. Define $\hat{\phi} = \lim_{N\to\infty}\sum_{i=1}^N a_i(f^0)/N$. We assume that λ_i takes only a finite number of values, $\lambda(1), \lambda(2), \ldots, \lambda(K)$. Let σ_k denote the limit fraction of $\lambda(k)$ among N firms when $N \to \infty$. Define $\mu = \xi_a \psi \mathrm{E}[\epsilon_i]\mathrm{E}[1/\lambda]$. Then we obtain a full characterization of g_N.

Proposition 3 (Production growth distribution for a general case) *Suppose that sequences B_i^0, λ_i^0, $i = 1, 2, \ldots$, are bounded and independent when i is randomly chosen, and λ_i takes a finite number of values. Suppose that we have an equilibrium f^0. A_i^0 is distributed i.i.d. across i so that s_i^0 has a cumulative distribution function F_s that satisfies $\lim_{h\to 0}(F_s(1) - F_s(1-h))/h = \psi$. Consider a perturbation $\log A_i^1 = \log A_i^0 + \epsilon_i/N$ where ϵ_i is i.i.d. across i, positive, and bounded. Then, when $\hat{\phi}\psi\mathrm{E}[b/\lambda] \leq 1$, Ng_N/c_0 asymptotically follows a moment generating function $e^{\mu(G(s)-1)}$ where $G(s)$ satisfies a functional equation:*

$$G(s) = \sum_{k=1}^{K} J_{x^0}((s + \psi\mathrm{E}[1/\lambda](G(s)-1))b(k))\sigma_k/(\lambda(k)\mathrm{E}[1/\lambda]). \tag{26}$$

Proof: See Appendix A.6.3.

By Proposition 3 we can compute any moment of the aggregate fluctuations. Moreover, the essential properties of the propagation are preserved from Proposition 1, namely, the N-times slower convergence and the infinite divisibility of the propagation distribution.

5 Conclusion

In this paper, we analyze a generic model of an (S, s) economy with finite agents where each agent follows a threshold adjustment policy. A simple supermodular game offers a concise framework of agents' decisions that underlies our reduced-form modeling. We derive an asymptotic distribution of propagation caused by a positive feedback effect across agents' policies. With homogeneous agents, we derive the closed-form distribution of the propagation. The distribution shows a slower convergence to a deterministic value than its counterpart in a smoothly-adjusting economy. Moreover, the distribution is skewed and heavy-tailed. The variance of the propagation is significantly larger than its smooth counterpart due to the slow convergence and the heavy tail, and hence contrasts the neutrality theorems on the (S, s) economy in which the threshold behavior does not cause significant aggregate fluctuations.

The distribution exhibits a phase transition depending on the size of the lumpiness relative to the size of an exogenous shock. The distribution is skewed and heavy-tailed, and slowly converging to a deterministic value when the lumpiness is larger than the shock, whereas it follows a normal distribution and converges as fast as the central limit theorem predicts when the lumpiness is overwhelmed by the shock. Applying this idea to the case when the shocks accumulate over time, we show that short-run fluctuations are characterized by the skewed and heavy-tailed distribution, whereas long-run fluctuations are characterized by the normal distribution. Furthermore, by utilizing the infinite divisibility of the propagation distribution, the equilibrium path can be approximated by a compound Poisson process for a short time horizon, whereas the process progressively converges to a normal process as the time horizon becomes longer.

The result of aggregate fluctuations is shown to obtain in a standard sectoral business cycle model that incorporates a general equilibrium setup. A key assumption added is the lumpy adjustment of production inputs. The model generates strong amplification effects of independent sectoral shocks which may explain business cycle phenomena. Finally we see the link between our model and the models of critical phenomena, in particular the self-organized criticality model. We suggest that the partial equilibrium phenomena such as the dynamics of industrial groups may exhibit a power-law distribution of fluctuations when the system is characterized by constant returns to scale, threshold behaviors, and strategic complementarity.

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A Appendix

A.1 Proof of Proposition 1

Consider the best response dynamics (3,4) for $u = 1, 2, \ldots, T$. Define $M_u = N(Q_N(x_u) - Q_N(x_{u-1}))$ for $u \geq 1$. Define m_u as the number of agents that increase x_i at u. Under Assumption 2, $M_u = m_u \phi \lambda$ holds. We first prove the following lemma.

Lemma 4 (Branching process in best response dynamics) *Under Assumptions 1 and 2, the process m_u, $u = 1, \ldots, T$, follows asymptotically as $N \to \infty$ a branching process where the number of initial parents m_1 follows a Poisson distribution with mean μ and the number of children each parent bears follows a Poisson distribution with mean $\phi\psi$.*

Proof: Define H_u as a set of agents i such that $x_{i,u} - x_{i,u-1} = \lambda$. Let c denote the upper bound of the support of ϵ_i. We define Condition U on a path m_v, $v = 1, \ldots, u-1$ for $u \geq 2$, as $(c/\lambda + \phi(\sum_{v=1}^{u-1} m_v))/N < 1$. U is a sufficient condition for $H_v \cup H_u = \emptyset$ for any $v < u$.

First, we examine the stochastic process m_u under U up to a finite step. The probability that an agent i belongs to H_1 is $\Pr(s_{i,0} + \epsilon_i/(N\lambda) \geq 1) = \int_0^c F_s(1) - F_s(1 - \epsilon_i/(N\lambda)) f(d\epsilon_i)$. Thus m_1 follows a binomial distribution with this probability and population N. Similarly, for $u \geq 2$, the probability that an agent $i \notin \bigcup_{v=1}^{u-1} H_v$ belongs to H_u is derived as follows. Define a short-hand $h_1 = (\epsilon_i/\lambda + \phi \sum_{v=1}^{u-1} m_v)/N$

and $h_2 = (\epsilon_i/\lambda + \phi \sum_{v=1}^{u-2} m_v)/N$.

$$\Pr(s_{i,u-1} + \phi m_{u-1}/N \geq 1 \mid \{m_v\}_{v=1}^{u-1}, \, i \notin \bigcup_{v=1}^{u-1} H_v) \tag{27}$$

$$= \frac{\Pr(s_{i,u-1} + \phi m_{u-1}/N \geq 1, \, i \notin \bigcup_{v=1}^{u-1} H_v \mid \{m_v\}_{v=1}^{u-1})}{\Pr(i \notin \bigcup_{v=1}^{u-1} H_v \mid \{m_v\}_{v=1}^{u-1})} \tag{28}$$

$$= \frac{\int_0^c F_s(1-h_2) - F_s(1-h_1) f(d\epsilon_i)}{\int_0^c F_s(1-h_2) f(d\epsilon_i)} \tag{29}$$

The first equality holds by the multiplication rule of conditional probabilities. The second equality holds because $i \notin \bigcup_{v=1}^{u-1} H_v$ is equivalent to $s_{i,u-1} = s_{i,0} + (\epsilon_i/\lambda + \phi \sum_{v=1}^{u-2} m_v)/N < 1$. Hence, m_u given $\{m_v\}_{v=1}^{u-1}$ for $u \geq 2$ under U follows a binomial distribution with probability (29) and population $N - \sum_{v=1}^{u-1} m_v$.

Next, we derive an asymptotic process of m_u up to a finite step. We showed that m_u follows a stochastic process which is finite with probability one up to a finite step. Hence, by construction, U is satisfied with probability one when $N \to \infty$ up to a finite step. The asymptotic mean of m_1 is simply: $\int_0^c F_s(1) - F_s(1 - \epsilon_i/(N\lambda)) f(d\epsilon_i) N \to \mathrm{E}[\epsilon_i]\psi/\lambda = \mu$ as $N \to \infty$. Hence m_1 asymptotically follows a Poisson distribution with mean μ. The asymptotic mean of m_u given m_{u-1} for $u \geq 2$ is derived similarly. Note that $h_1 \to 0$ and $h_2 \to 0$ as $N \to \infty$ for any finite path of m_v. Then,

$$\frac{\int_0^c F_s(1-h_2) - F_s(1-h_1) f(d\epsilon_i)}{\int_0^c F_s(1-h_2) f(d\epsilon_i)} (N - \sum_{v=1}^{u-1} m_v)$$
$$\to \quad (-\psi(\mathrm{E}[\epsilon_i]/\lambda + \phi \sum_{v=1}^{u-2} m_v) + \psi(\mathrm{E}[\epsilon_i]/\lambda + \phi \sum_{v=1}^{u-1} m_v)) = \phi\psi m_{u-1}. \tag{30}$$

Hence m_u given m_{u-1} asymptotically follows a Poisson distribution with mean $\phi\psi m_{u-1}$. Since a Poisson distribution is infinitely divisible, the process m_u given m_1 follows a branching process whose step distribution follows a Poisson with mean $\phi\psi$.

The stopping time T is finite with probability one, since the mean number of children born by a parent is $\phi\psi \leq 1$ by Assumption 2 (Harris (1989)). Hence U is satisfied by an entire path m_v, $v = 1, 2, \ldots, T$, with probability one. □

The sum of the branching process $\sum_{v=1}^{T} m_v$ given $m_1 = l$ is known to follow a Borel-Tanner distribution (Kingman, 1993, page 68), $\Pr(\sum_{u=1}^{T} m_u = w \mid m_1 = l) = (l/w) e^{-\phi w} (\phi w)^{w-l}/(w-l)!$, for $w = l, l+1, \ldots$. By Lemma 4, m_1 follows a Poisson distribution with mean μ. Thus $\sum_{v=1}^{T} m_v$ follows a compound Poisson distribution with Poisson mean μ and a Borel-Tanner for $m_1 = 1$. Calculating the compound distribution, we obtain the desired distribution function (7). This distribution is infinitely divisible since this is a compound Poisson distribution. Equation (8) is obtained by applying a Stirling's formula $w! = \sqrt{2\pi} w^{w+0.5} e^{-w}$ to (7) for large w. □

A.2 Proof of Lemma 2

Using the definition (2) of s_i, we obtain:

$$s_i^1 = (s_i^0 + ((\epsilon_i + Q_N(x^1) - Q_N(x^0))/\lambda_i)(\text{mod } 1))(\text{mod } 1). \tag{31}$$

By assumption, s_i^0 and the second term in the right hand side is asymptotically independent. The convolution of a uniform distribution and an arbitrary distribution on a circle yields a uniform distribution (Feller, 1966, page 64). The asymptotic independence of s_i and s_j is shown as follows. Equation (31) is rewritten as $s_i^1 = (r_i + r_c)(\text{mod } 1)$ where $r_i \equiv (s_i^0 + \epsilon_i/\lambda_i)(\text{mod } 1)$ and $r_c \equiv ((Q_N(x^1) - Q_N(x^0))/\lambda_i)(\text{mod } 1)$. Note that r_i follows a uniform distribution independently across i. Then for any fixed r_c, (s_i, s_j) follows a uniform distribution over $[0, 1)^2$. Since r_i and r_c are asymptotically independent, (s_i, s_j) is asymptotically distributed uniformly over $[0, 1)^2$ as $N \to \infty$. □

A.3 Proof of Lemma 3

By the definition of s_i (2), we have:

$$s_{i,N^p} = (s_{i,0} + ((Q_N(x_{N^p}) - Q_N(x_0))/\lambda_i)(\text{mod } 1) + (\sum_{t=0}^{N^p-1} \epsilon_{i,t}/(N\lambda_i))(\text{mod } 1))(\text{mod } 1) \tag{32}$$

The accumulated sum of $\epsilon_{i,t}/(N\lambda_i)$ asymptotically follows a normal distribution with a diverging variance when $p > 2$ by the central limit theorem. Hence its modulo one converges in distribution to a uniform distribution over $[0, 1)$ (Feller, 1966, page 62). By assumption, the summation term and other terms are asymptotically independent. Hence the entire sum modulo one also converges in distribution to a distribution uniform over a unit interval. As for independence, the same argument as in Appendix A.2 applies. □

A.4 Derivation of equilibrium in Section 4

In this section we solve for fundamental relations that characterize an equilibrium. Define a price index P as: $P \equiv (\sum_{k=1}^{N} p_k^{1-\sigma}/N)^{1/(1-\sigma)}$. Demand for a component of a CES composite good is known to be a linear function of the composite good demand. Thus we obtain the following demand function for good i: $y_i = (p_i/P)^{-\sigma}(C + \sum_j Z_j + \sum_j f_j/B_j)/N$. The elasticity of demand for i is thus equal to the elasticity of substitution among inputs, σ.

By using the demand function, the factor demand functions for L_i and Z_i for a given f_i are derived as in a usual maximization problem of a monopolist. Substituting back the optimal rules for L_i and Z_i into the production function, we obtain a relationship: $y_i = D_0(A_i f_i^{(1-\alpha)\theta} p_i^{\alpha\theta+\gamma} P^{-\alpha\theta})^{1/\xi_0}$ where $D_0 \equiv ((1-1/\sigma)^{\alpha\theta+\gamma}(\alpha\theta)^{\alpha\theta}\gamma^{\gamma})^{1/\xi_0}$ and $\xi_0 = 1-\alpha\theta-\gamma$. Then we can solve the prices as: $p_i/P = (A_i f_i^{(1-\alpha)\theta})^{-1/\xi_1} S(A,f)^{(1-\alpha)\theta/\xi_1}$ where $\xi_1 \equiv \sigma\xi_0 - \xi_0 + 1$ and,

$$\phi \equiv \frac{(1-\alpha)\theta((1-\sigma)\gamma + (1+\nu)/(\rho+\nu))}{((\sigma-1)(1-\gamma-\theta)+1)((1-\alpha\theta-\gamma)(1+\nu)/(\rho+\nu)+\gamma)}. \tag{33}$$

By substituting y_i in the supply function and the price formula into the monopolist's problem (17), a reduced problem is obtained to determine the optimal level of plants $f_i \in \arg\max_{f_i} \pi(f_i)$ such that:

$$\pi(f_i) = D_0 D_1 (A_i f_i^{(1-\alpha)\theta})^{(\sigma-1)/\xi_1} S(f)^{(1-\alpha)\theta/(\xi_0\xi_1)} P^{(1-\alpha\theta)/\xi_0} - P f_i/B_i \tag{34}$$

and $D_1 \equiv 1-(1-1/\sigma)(\alpha\theta+\gamma)$. The target function (34) is strictly concave in f_i when B_i is large enough and the effect of f_i on $S(f)$ is negligible for the monopolist. Thus the maximizer f_i exists in a closed set $\{1, e^{\pm\lambda_i}, e^{\pm 2\lambda_i}, \ldots\}$. Also there exists a unique real number χ_i such that $\pi(\chi_i) = \pi(\chi_i/e^{\lambda_i})$ and the optimal f_i satisfies $\chi_i/e^{\lambda_i} < f_i \le \chi_i$. This means that at the optimal f_i there is no incentive to increase or decrease f_i by e^{λ_i}. The last inequality turns into the following:

$$0 \le (D_i' + \xi_a \log A_i + \xi_b \log B_i + \xi_p \log P + \xi_s \log S(f) - \log f_i)/\lambda_i < 1 \tag{35}$$

where $\xi_2 \equiv \xi_1 - (1-\alpha)\theta(\sigma-1)$, $\xi_a \equiv (\sigma-1)/\xi_2$, $\xi_b \equiv \xi_1/\xi_2$, $\xi_p \equiv \gamma\xi_1/(\xi_0\xi_2)$, $\xi_s \equiv (1-\alpha)\theta/(\xi_0\xi_2)$, and $D_i' \equiv (\xi_1/\xi_2)\log(D_0 D_1(1-e^{-\lambda_i(1-\alpha)\theta(\sigma-1)/\xi_1})/(1-e^{-\lambda_i}))$.

This inequality shows that the optimal number of plants is a function of the price level P and the aggregate supply capacity $S(f)$. At the same time, the inequality implies an inaction region for f_i in the sense that f_i does not respond to a perturbation in A_i, B_i, P, or $S(f)$ as long as the perturbation is small enough for the inequality to continue to hold at the original optimum level.

The price level P (and the real wage $1/P$) is equal to the ratio of marginal utilities of commodities and labor: $P = -U_C/U_L = C^{-\rho}/L^{\nu}$. Note that $\sum_k p_k y_k = PY$. By labor demand, $L = (1-1/\sigma)\gamma PY$ holds. On the other hand, by the consumer's budget constraint and derived demand functions,

$$C \approx (1-(1-1/\sigma)\theta)Y \tag{36}$$

holds approximately. The approximation method is to evaluate the cost of plants at the level of the frictionless counterpart. Thus the approximation is exact when the discreteness $e^{\lambda_i} - 1$ is vanishingly small for all i. Numerical calculations show that the approximation error relative to total consumption is $(0.000, 0.004, 0.014, 0.03, 0.05)$ percent for various common lumpiness size $e^{\lambda} = (1.01, 1.05, 1.10, 1.15, 1.20)$ under the parameter set $\alpha = 0.25$, $\gamma = 0.5$, $\theta = 1$, and $\sigma = 2.5$. Since the magnitude of aggregate fluctuations we generate is about 1 percent in standard deviation, we regard this error as negligible. Henceforth we regard that it is accurate enough. Then we obtain:

$$P = D_2 Y^{-\tilde{\rho}} \tag{37}$$

where $\tilde{\rho} \equiv (\rho + \nu)/(1 + \nu)$ and $D_2 \equiv (1 - (1 - 1/\sigma)\theta)^{-\rho/(1+\nu)}/((1 - 1/\sigma)\gamma)^{\nu/(1+\nu)}$. On the other hand, the good supply y_i is a function of P, hence aggregate supply Y is a function of P. Solving for P, we obtain:

$$P = D_3 S(f)^{-(1-\alpha)\theta/(\xi_0/\tilde{\rho}+\gamma)} \tag{38}$$

where $D_3 \equiv (D_0 D_2^{-1/\tilde{\rho}} N)^{-\xi_0/(\xi_0/\tilde{\rho}+\gamma)}$. Substituting P out of (35), we obtain an equilibrium condition for f_i,

$$0 \leq (D_i + \xi_a \log A_i + \xi_b \log B_i + \phi \log S(f) - \log f_i)/\lambda_i \leq 1 \tag{39}$$

where $D_i \equiv D_i' + (\gamma\xi_1/\xi_0\xi_2) \log D_3$.

A.5 Proof of Proposition 2

By (37), (38), and the definition of g_N, we obtain that:

$$N g_N / c_0 = N\phi(\log S(f^1) - \log S(f^0)) \tag{40}$$

By a Taylor series expansion of $\log S(f)$ around f, we obtain $N\phi(\log S(f_u) - \log S(f_{u-1})) \to \phi\lambda m_u$ when $N \to \infty$ and then $\sigma \to 1$, where m_u is the number of firms that adjust at u (see the reference in the proof of Proposition 3 for a complete derivation). Since $\xi_1 \to 1$ and $\xi_2 \to 1$ when $\sigma \to 1$, m_1 asymptotically follows a Poisson distribution with mean μ. Hence Proposition 1 applies. □

A.6 Generalized model and proof of Proposition 3

For Proposition 1, Assumption 2 assumed the homogeneity of λ_i and the symmetry and linearity of $Q_N(x)$ in x. Our result can be extended to the case with heterogeneous agents and non-linear $Q_N(x)$. We show the propagation result in this generalized setup in Lemma 5.

A.6.1 Generalized model

We allow λ_i to be heterogeneous with a finite number of types K that take values $\lambda(1), \lambda(2), \ldots, \lambda(K)$. We call an agent with $\lambda(k)$ "type-k." Let σ_k denote the limit fraction of type-k agents among all the agents when $N \to \infty$. Consider two sequences of real numbers a_i and λ_i, $i = 1, 2, \ldots$. We assume that a_i and λ_i are mutually independent when i is drawn randomly. Define a function $b(\lambda_i)$ and let $b(k)$ denote $b(\lambda(k))$. We write the limit of the averages of $1/\lambda_i$ and $b(\lambda_i)/\lambda_i$ for $i = 1, 2, \ldots, N$ when $N \to \infty$ by using an expectation operator as $\mathrm{E}[1/\lambda] \equiv \sum_{k=1}^{K} \sigma_k/\lambda(k)$ and $\mathrm{E}[b/\lambda] \equiv \sum_{k=1}^{K} \sigma_k b(\lambda(k))/\lambda(k)$, respectively.

The relaxed assumption is as follows.

Assumption 3 (Generalization) *For any finite set H, a sequence of bounded functions Q_N satisfies $N(Q_N(\{x_i+\lambda_i\}_{i\in H}, x_{-H}) - Q_N(x)) \to \sum_{i\in H} a_i(x)b(\lambda_i)$ as $N \to \infty$. A sequence λ_i, $i = 1, 2, \ldots$, takes a finite number of values. For each x, a sequence $a_i(x)$, $i = 1, 2, \ldots$, is bounded and satisfies $\sum_{i=1}^{N} a_i(x)/N \to \phi(x)$ where $\phi(x)\psi\mathrm{E}[b/\lambda] \leq 1$. The pair $(a_i(x), \lambda_i)$ is mutually independent for each x when i is randomly drawn.*

Assumption 3 allows a heterogeneous effect, $a_i(x)b(\lambda_i)$, of an adjustment of x_i on $Q(x)$. A simple example of such a $Q_N(x)$ is $\sum_i a_i x_i/N$, which includes the homogeneous case as a special case when $a_i = \phi$. Assumption 3 also allows the effect to depend on λ_i and x. Dependence on x is permissible since an asymptotic distribution of perturbation is determined only by the local characteristics of $Q_N(x)$ at x^0. An example of such a nonlinear aggregator $Q_N(x)$ is a CES-type function that we use in Section 4.

Define $\mu = \psi\mathrm{E}[\epsilon_i]\mathrm{E}[1/\lambda]$. Define $J_x(\cdot)$ as a moment generating function of $a_i(x)$ when i is randomly drawn. Under the relaxed assumption, we obtain the following distribution of the propagation size:

Lemma 5 (Propagation distribution in a general case) *Under Assumptions 1 and 3, the normalized propagation size $N(Q_N(x^1) - Q_N(x^0))$ asymptotically follows a moment generating function $e^{\mu(G(s)-1)}$ where $G(s)$ satisfies a functional equation:*

$$G(s) = \sum_{k=1}^{K} J_{x^0}((s + \psi\mathrm{E}[1/\lambda](G(s) - 1))b(k))\sigma_k/(\lambda(k)\mathrm{E}[1/\lambda]). \tag{41}$$

The propagation distribution is infinitely divisible.

Proof is shown in the next section. From the formula above, one can calculate any moment of the propagation size $Q_N(x^1) - Q_N(x^0)$ asymptotically. For example, its mean is $(\mathrm{E}[\epsilon_i]/N)\mathrm{E}[b/\lambda]\phi\psi/(1 - \phi\psi\mathrm{E}[b/\lambda])$ and its variance is

$(\mathrm{E}[\epsilon_i]/N^2)\mathrm{E}[b^2/\lambda]\mathrm{E}[a_i^2(x^0)]\psi/(1-\phi\psi\mathrm{E}[b/\lambda])^3$. The mean and variance in the homogeneous model can be reproduced by substituting $a_i = \phi$ and $b(\lambda_i) = \lambda$.

The aggregate behavior of the economy's smooth counterpart is analogous to the homogeneous case. Consider a linear case $Q_N(x) = \sum_i a_i x_i/N$. If agents adjust smoothly as (1), then $x_i^1 - x_i^0 = Q_N(x^1) - Q_N(x^0) + \epsilon_i/N$ holds. Then we obtain: $\sum_{i=1}^N a_i(x_i^1 - x_i^0) = (\sum_{i=1}^N a_i/N)N(Q_N(x^1) - Q_N(x^0)) + \sum_{i=1}^N a_i\epsilon_i/N$. Thus, the normalized propagation size is $N(Q_N(x^1) - Q_N(x^0)) = (\sum_{i=1}^N a_i\epsilon_i/N)/(1-\sum_{i=1}^N a_i/N)$. Hence, $N^{1.5}(Q_N(x^1) - Q_N(x^0) - \mathrm{E}[\epsilon_i]\phi/(1-\phi)N)$ follows a normal distribution with mean zero and variance $\mathrm{Var}[a_i\epsilon_i]/(1-\phi)^2$. The variance of the propagation size converges to zero as fast as N^{-3}. The homogeneous case is again a particular case of this result.

By the form of the moment generating function, the normalized propagation size follows a compound Poisson distribution with Poisson mean μ and a random variable that follows a moment generating function $G(s)$. Thus, the distribution function is infinitely divisible. Therefore, the time series implication derived for the homogeneous case applies to the heterogeneous case. Suppose that e_i^t evolves as a stochastic process with independent and positive increments. Then we can define a sequence of static equilibria. For a fixed large N, suppose that ϵ_i/N is equivalent to an increment of e_i for a unit time horizon $e_i^1 - e_i^0$. Then the sequence of static equilibria is approximated by a compound Poisson process with hazard rate μ and a random variable that follows $G(s)$ for a time horizon less than the unit time.

A.6.2 Proof of Lemma 5

Define $M_u = N(Q_N(x_u) - Q_N(x_{u-1}))$ for $u \geq 1$. The normalized propagation size is $N(Q_N(x^1) - Q_N(x^0)) = \sum_{v=1}^T M_v$. Define $H_u(k)$ as a set i such that $x_{i,u} - x_{i,u-1} = \lambda(k)$, namely type-$k$ agents that increase x_i at u. Define $m_u(k)$ as the number of elements in $H_u(k)$. Define $H_u = \bigcup_{k=1}^K H_u(k)$ and $m_u = \sum_{k=1}^K m_u(k)$. Define a fraction of k-type agents for each N as σ_k^N. Let c denote the upper bound of the support of ϵ_i.

Define Condition U on a path M_v, $v = 1, 2, \ldots, u-1$, for $u \geq 2$ as:

$$(c + \sum_{v=1}^{u-1} M_v)/(N\underline{\lambda_i}) < 1. \tag{42}$$

Condition U is a sufficient condition for $H_u \cup H_v = \emptyset$ to hold for any $v < u$, by the construction of the best response dynamics.

Let us examine the distributions of $m_1(k)$. The probability that a type-k agent i belongs to $H_1(k)$ is:

$$\Pr(s_{i,0} + \epsilon_i/(N\lambda_i) \geq 1 \mid \lambda_i = \lambda(k)) = \int_0^c F_s(1) - F_s(1 - \epsilon_i/(N\lambda(k)))f(d\epsilon_i). \tag{43}$$

Thus $m_1(k)$ follows a binomial distribution with probability (43) and population $\sigma_k^N N$.

Similarly, the probability that a type-k agent $i \notin \bigcup_{v=1}^{u-1} H_v(k)$ belongs to $H_u(k)$ for any $u \geq 2$ is expressed as follows. Define a short-hand $h_1 = (\epsilon_i + \sum_{v=1}^{u-1} M_v)/(N\lambda(k))$ and $h_2 = (\epsilon_i + \sum_{v=1}^{u-2} M_v)/(N\lambda(k))$.

$$\Pr(s_{i,u-1} + M_{u-1}/(N\lambda_i) \geq 1 \mid \{M_v\}_{v=1}^{u-1},\, i \notin \bigcup_{v=1}^{u-1} H_v(k), \lambda_i = \lambda(k)) \tag{44}$$

$$= \frac{\Pr(s_{i,u-1} + M_{u-1}/(N\lambda_i) \geq 1,\, i \notin \bigcup_{v=1}^{u-1} H_v(k) \mid \{M_v\}_{v=1}^{u-1}, \lambda_i = \lambda(k))}{\Pr(i \notin \bigcup_{v=1}^{u-1} H_v(k) \mid \{M_v\}_{v=1}^{u-1}, \lambda_i = \lambda(k))} \tag{45}$$

$$= \frac{\int_0^c F_s(1-h_2) - F_s(1-h_1) f(d\epsilon_i)}{\int_0^c F_s(1-h_2) f(d\epsilon_i)} \tag{46}$$

The first equality holds by the multiplication rule of conditional probabilities. The second equality holds because $i \notin \bigcup_{v=1}^{u-1} H_v$ is equivalent to $s_{i,u-1} = s_{i,0} + (\epsilon_i + \sum_{v=1}^{u-2} M_v)/(N\lambda_i) < 1$. Thus, $m_u(k)$ given $\{M_v, m_v\}_{v=1}^{u-1}$ for $u \geq 2$ which satisfies Condition U follows a binomial distribution with probability (46) and population $\sigma_k^N N - \sum_{v=1}^{u-1} m_v(k)$.

M_u given m_u follows a bounded random variable $N(Q_N(x_u) - Q_N(x_{u-1}))$. Combining all above, (M_u, m_u) follow a stochastic process which is finite with probability one up to a finite step. Hence Condition U is satisfied with probability one when $N \to \infty$ up to a finite step. Let us derive an asymptotic process of (M_u, m_u) up to a finite step. First, the asymptotic mean of $m_1(k)$ is calculated as:

$$\int_0^c ((F_s(1) - F_s(1 - \epsilon_i/(N\lambda(k))))/(\epsilon_i/(N\lambda(k))))(\epsilon_i/(N\lambda(k)))f(d\epsilon_i)\sigma_k^N N$$
$$\to \psi \mathrm{E}[\epsilon_i]\sigma_k/\lambda(k) \tag{47}$$

when $N \to \infty$. Hence $m_1(k)$ asymptotically follows a Poisson distribution with mean (47). Next we calculate the asymptotic mean of $m_u(k)$ given M_{u-1}. Note that $h_1 \to 0$ and $h_2 \to 0$ as $N \to \infty$ for any finite path of M_v.

$$\frac{\int_0^c F_s(1-h_2) - F_s(1-h_1) f(d\epsilon_i)}{\int_0^c F_s(1-h_2) f(d\epsilon_i)} (\sigma_k^N N - \sum_{v=1}^{u-1} m_v(k))$$
$$\to (-\psi(\mathrm{E}[\epsilon_i] + \sum_{v=1}^{u-2} M_v)/\lambda(k) + \psi(\mathrm{E}[\epsilon_i] + \sum_{v=1}^{u-1} M_v)/\lambda(k))\sigma_k \tag{48}$$
$$= M_{u-1}\psi\sigma_k/\lambda(k) \tag{49}$$

Hence $m_u(k)$ given M_{u-1} asymptotically follows a Poisson distribution with mean (49). Its moment generating function is:

$$\mathrm{E}[e^{s m_u(k)} \mid M_{u-1}] = e^{(M_{u-1}\psi\sigma_k/\lambda(k))(e^s - 1)}. \tag{50}$$

On the other hand, M_u given $\{m_u(k)\}_{k=1}^K$ asymptotically follows a convolution for $k = 1, 2, \ldots, K$ of $m_u(k)$-fold convolutions of $J_{x^0}(sb(k))$. This is shown as follows. By Assumption 3, $N(Q_N(x_u) - Q_N(x^0)) \to \sum_{i \in \bigcup_{v=1}^{u} H_v} a_i(x^0)b(\lambda_i)$ and $N(Q_N(x_{u-1}) - Q_N(x^0)) \to \sum_{i \in \bigcup_{v=1}^{u-1} H_v} a_i(x^0)b(\lambda_i)$ as $N \to \infty$. Thus:

$$M_u = N(Q_N(x_u) - Q_N(x_{u-1})) \to \sum_{i \in H_u} a_i(x^0)b(\lambda_i) = \sum_{k=1}^{K} \sum_{i \in H_u(k)} a_i(x^0)b(k). \tag{51}$$

We obtain an asymptotic moment generating function:

$$\mathrm{E}[e^{sM_u} \mid \{m_u(k)\}_{k=1}^K] = \prod_{k=1}^{K} J_{x^0}^{m_u(k)}(sb(k)). \tag{52}$$

Let us suppress the subscript x^0 in J_{x^0} henceforth.

A vector of Poisson random variables $(m_u(1), \ldots, m_u(K))$ given its sum m_u follows a multinomial distribution with a probability vector $(\sigma_1/(\lambda(1)\mathrm{E}[1/\lambda]), \ldots, \sigma_K/(\lambda(K)\mathrm{E}[1/\lambda]))$ and population m_u (Kingman, 1993, page 7). From this fact and (52) we obtain:

$$\begin{aligned} &\mathrm{E}[e^{sM_u} \mid m_u] \\ &= \sum_{\{(m_u(1),\ldots,m_u(K)\}} \left(\prod_k J^{m_u(k)}(sb(k))\right) \frac{m_u!}{m_u(1)! \cdots m_u(K)!} \prod_k \left(\frac{\sigma_k/\lambda(k)}{\mathrm{E}[1/\lambda]}\right)^{m_u(k)} &(53) \\ &= \sum \frac{m_u!}{m_u(1)! \cdots m_u(K)!} \prod_k \left(\frac{J(sb(k))\sigma_k/\lambda(k)}{\sum_k J(sb(k))\sigma_k/\lambda(k)}\right)^{m_u(k)} \left(\frac{\sum_k J(sb(k))\sigma_k/\lambda(k)}{\mathrm{E}[1/\lambda]}\right)^{m_u} &(54) \\ &= \left(\sum_k J(sb(k))\sigma_k/(\lambda(k)\mathrm{E}[1/\lambda])\right)^{m_u} &(55) \end{aligned}$$

Combining (50) and (55), we obtain that:

$$\begin{aligned} \mathrm{E}[e^{sm_u} \mid m_{u-1}] &= \mathrm{E}[\mathrm{E}[e^{sm_u} \mid M_{u-1}] \mid m_{u-1}] &(56) \\ &= \mathrm{E}[e^{(e^s-1)M_{u-1}\psi \sum_k (\sigma_k/\lambda(k))} \mid m_{u-1}] &(57) \\ &= (\sum_k J(\psi \mathrm{E}[1/\lambda](e^s - 1)b(k))\sigma_k/\lambda(k)\mathrm{E}[1/\lambda])^{m_{u-1}} &(58) \end{aligned}$$

The generating function above is divisible into m_{u-1} independent generating functions that have mean $\phi\psi\mathrm{E}[b/\lambda] \le 1$. Hence, for any finite V, $\{m_v\}_{v=1}^V$ given m_1 asymptotically follows a branching process where there are m_1 initial parents and the number of children born by each parent has mean less than or equal to one. Its extinction time is finite with probability one (Harris (1989)). Also m_1 is finite with probability one. Thus the stopping time T is finite with probability one. Hence Condition U is satisfied by an entire path M_v, $v = 1, 2, \ldots, T$, with probability one.

Finally we examine the asymptotic distribution of M_u given M_{u-1}. Its moment generating function is derived by combining (50) and (52) as:

$$
\begin{aligned}
\mathrm{E}[e^{sM_u} \mid M_{u-1}] &= \mathrm{E}[\mathrm{E}[e^{sM_u} \mid \{m_u(k)\}_k] \mid M_{u-1}] && (59)\\
&= \prod_k \mathrm{E}[J^{m_u(k)}(sb(k)) \mid M_{u-1}] && (60)\\
&= \prod_k e^{(M_{u-1}\psi\sigma_k/\lambda(k))(J(sb(k))-1)} && (61)\\
&= e^{M_{u-1}\psi\mathrm{E}[1/\lambda]\left((\sum_k J(sb(k))\sigma_k/\lambda(k))/\mathrm{E}[1/\lambda]-1\right)} && (62)
\end{aligned}
$$

The form of the moment generating function (62) indicates that M_u follows a compound Poisson distribution with Poisson distribution with mean $M_{u-1}\psi\mathrm{E}[1/\lambda]$ and a random variable that follows a moment generating function $\sum_k J(sb(k))\sigma_k/(\lambda(k)\mathrm{E}[1/\lambda])$.

Define $G(s)$ as an asymptotic moment generating function of $\sum_{u=1}^{T} M_u$ given $m_1 = 1$. Since a compound Poisson distribution is infinitely divisible, we can divide the subsequent propagation after $m_1 = 1$ into m_2 parts where each part is the random variable $\sum_{u=2}^{T} M_u$ given $m_2 = 1$. Thus $G(s)$ satisfies a functional equation $G(s) = \mathrm{E}[e^{sM_1}G^{m_2}(s) \mid m_1 = 1]$. Using (50) and (55), we obtain:

$$
\begin{aligned}
G(s) &= \mathrm{E}[e^{sM_1}\mathrm{E}[G^{m_2}(s) \mid M_1] \mid m_1 = 1] && (63)\\
&= \mathrm{E}[e^{M_1(s+\psi\mathrm{E}[1/\lambda](G(s)-1))} \mid m_1 = 1] && (64)\\
&= \sum_k J((s + \psi\mathrm{E}[1/\lambda](G(s) - 1))b(k))\sigma_k/(\lambda(k)\mathrm{E}[1/\lambda]) && (65)
\end{aligned}
$$

On the other hand, $m_1(k)$ follows a Poisson with mean (47). Thus m_1 follows a Poisson with μ. Hence $\mathrm{E}[e^{s\sum_{u=1}^{T} M_u}] = \mathrm{E}[G^{m_1}(s)] = e^{\mu(G(s)-1)}$. This is a compound Poisson distribution with Poisson mean μ and a random variable that follows $G(s)$. Since $e^{(\mu/n)(G(s)-1)}$ is a moment generating function for any integer n, the compound Poisson distribution is infinitely divisible. □

A.6.3 Proof of Proposition 3

Recall that:

$$
Ng_N/c_0 = N\phi(\log S(A^1, f^1) - \log S(A^0, f^0)) \quad (66)
$$

Let us first show that the direct effect of perturbation, $N\phi(\log S(A^1, f^0) - \log S(A^0, f^0))$, is vanishingly small. Let us define, for each realization of A^1,

$$a_i^N(f) \equiv \frac{\phi(A_i^1 f_i^{(1-\alpha)\theta})^{(\sigma-1)/\xi_1}}{\sum_{j=1}^N (A_j^1 f_j^{(1-\alpha)\theta})^{(\sigma-1)/\xi_1}/N} \tag{67}$$

By a Taylor series expansion of $\log S(A^1, f^0)$ around A^0, we obtain:

$$\begin{aligned} &N^{1.5}\phi(\log S(A^1, f^0) - \log S(A^0, f^0)) = \\ &\quad (\sqrt{N}\phi/((1-\alpha)\theta)) \sum_{i=1}^N \left(a_i^N (e^{\epsilon_i/N} - 1) + (a_i^N + (a_i^N)^2(\sigma-1)/(N\xi_1))(e^{\hat{\epsilon}_i/N} - 1)^2 \right) \end{aligned} \tag{68}$$

for some $\hat{\epsilon}_i \in (0, \epsilon_i)$ where a_i^N is evaluated at f^0. Also, if we define $y_i = N(e^{\epsilon_i/N} - 1)$, we can express ϵ_i by a Taylor expansion around $y_i = 0$ as $y_i - \hat{y}_i^2/N$ for some $\hat{y}_i \in (0, y_i)$. The second term vanishes as $N \to \infty$. By combining these, we obtain for large N that (68) is equal to $(N^{0.5}\phi/((1-\alpha)\theta)) \sum_i (a_i \epsilon_i/N + (a_i + a_i^2(\sigma-1)/N\xi_1)(\epsilon_i/N)^2)$. The second term degenerates to zero as $N \to \infty$. The first term converges to a normal distribution with finite variance by the central limit theorem. Noting that the normalization factor in (68) is $N^{1.5}$, we obtain that $N\phi(\log S(A^1, f^0) - \log S(A^0, f^0))$ degenerates to a deterministic zero as $N^{-0.5}$. Thus we can concentrate on the propagation effect evaluated at A^1, namely, $N\phi(\log S(A^1, f^1) - \log S(A^1, f^0))$. We suppress A^1 in S henceforth.

We next prove by induction that, for $n = 2, 3, \ldots$,

$$\partial^n N\phi \log S(f)/\partial f_i^n = (c_1 - 1)(c_1 - 2)\cdots(c_1 - n + 1)a_i^N(f)/f_i^n + O(1/N) \tag{69}$$

where $O(1/N)$ is here defined as a finite differentiable function of f_i divided by N and $c_1 = (1-\alpha)\theta(\sigma - 1)/\xi_1$. By taking direct derivatives of $\phi \log S(f)$, we have:

$$\begin{aligned} \partial N\phi \log S(f)/\partial f_i &= a_i^N(f)/f_i && (70) \\ \partial^2 N\phi \log S(f)/\partial f_i^2 &= (c_1 - 1)a_i^N(f)/f_i^2 - c_1(a_i^N(f)/f_i)^2/(N\phi). && (71) \end{aligned}$$

Thus our supposition holds for $n = 2$. Suppose it holds for n. Then for $n + 1$,

$$\begin{aligned} &\partial^{n+1} N\phi \log S(f)/\partial f_i^{n+1} && (72) \\ &= (\partial/\partial f_i)(\partial^n N\phi \log S(f)/\partial f_i^n) \\ &= (\partial/\partial f_i)(c_1 - 1)(c_1 - 2)\cdots(c_1 - n + 1)a_i^N(f)/f_i^n + O(1/N) \\ &= (c_1 - 1)(c_1 - 2)\cdots(c_1 - n + 1)((c_1 - n)a_i^N(f)/f_i^{n+1} - c_1(a_i^N(f))^2/(f_i^{n+1}N\phi)) \\ &\quad + O(1/N) \end{aligned}$$

Thus we verify our supposition.

Using the assumption $\theta + \gamma < \sigma/(\sigma - 1)$, we obtain $0 < c_1 < 1$. Thus the term $(c_1-1)(c_1-2)\cdots(c_1-n+1)(e^{\lambda}-1)^n/n!$ converges to zero as $n \to \infty$ for $0 < \lambda < \log 2$. By applying the above result for a Taylor series expansion of $N\phi \log S(f')$ around f where $f_i' = f_i e^{\lambda_i}$ for $i \in H$ and $f_i' = f_i$ for $i \notin H$ for any finite H, we obtain:

$$N\phi(\log S(f') - \log S(f)) = \sum_{i \in H} a_i^N(f) b(\lambda_i) + O(1/N) \tag{73}$$

where

$$b(\lambda_i) = e^{\lambda_i} - 1 + \sum_{n=2}^{\infty} (c_1 - 1)(c_1 - 2)\cdots(c_1 - n + 1)(e^{\lambda_i} - 1)^n/n! \tag{74}$$

The series is absolutely convergent, since $|(c_1 - 1)(c_1 - 2)\cdots(c_1 - n + 1)/n!| < 1$ for any $n \geq 2$. Hence, the aggregator function $\phi \log S(f)$ satisfies Assumption 3.

Only difference from the setup in Lemma 5 is that a_i^N and s_i^0 are not independent since A_i^1 and s_i^0 are correlated. Under the supposition that the normalized propagation size is finite with probability one, however, A_i^1 given that agent i adjusts in the best response dynamics converges to a deterministic value $\hat{A}_i^1$ as $N \to \infty$, since s_i^0 of the agents who adjust are concentrated on the vicinity of $s_i^0 = 1$. Also, a change in f_j for a finite set of j by a multiplication of e^{λ_j} does not affect the denominator of a_i^N in (67) at the limit of N. Hence, J defined as the moment generating function of $a_i(f^0)$ in (24) is the asymptotic distribution of M_u given $m_u = 1$ for any step u in the best response dynamics. The rest of the proof proceeds as in Appendix A.6.2. □

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