The Existence and Optimality of Equilibrium

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1 Introduction

These notes quickly survey two approaches to the existence. The first approach works with excess demand, while the second works with primitives, such as preferences, endowments, and the like, from which excess demand could be derived.

2 Excess Demand

For this approach in some detail, see Chapter 2 of Arrow and Hahn. Consider first an exchange economy. Each of I individuals has an endowment of L commodities, ω_i for individual i. Were individuals utility maximizers, we would derive a demand function or correspondence, $d_i(p)$, for the L commodities, which depends upon the vector p of market prices. **Excess demand** for individual n is $z_i(p) = d_i(p) - e_i$, the excess of trader i's demand over his supply. Aggregate excess demand is then $Z(p) = \sum_i z_i(p)$. Equilibrium can be defined in terms of excess demand:

Definition 1. *A* ***competitive equilibrium*** *is a price vector $p \gg 0$ such that $Z(p) = 0$.*

Another definition one often sees is:

Definition 2. *A* ***competitive equilibrium*** *is a price vector $p \geq 0$ such that $Z(p) \leq 0$ and $p \cdot Z(p) = 0$.*

The difference between the two is a complementary slackness condition. In the first, excess demand for each good must be 0. In the second, excess demand for a good could be negative, but then its price must be 0. We are going to build a model in this section in which the equilibrium price will be strictly positive, that is, all goods will be demanded in equilibrium. It is a good exercise to consider how to relax the assumptions that are to follow in such a way as to require the complementary slackness condition.

It is important to understand that while excess demand could be derived from utility maximization, it need not be. We could just think of excess demand as arbitrary behavior rules. If aggregate excess demand is sufficiently regular, an equilibrium (either definition) will exist. We will define demand for consumer i formally as a function

$$d_i : \mathbf{R}^{\mathbf{l}}_{++} \to \mathbf{R}_+.$$

The phrase "sufficiently regular" means:

A.1. $Z(p)$ *is homogeneous of degree 0.*

If choice is defined on budget sets, then this axiom must hold because the description of a given budget set is invariant to changes in scale of the prices which define it.

Because of homogeneity, we can *normalize* prices, which is to say that we can choose a scale for prices. It is convenient to choose a scale such that $\sum_l p_l = 1$. Let $\Delta = \{p \in \mathbf{R}^{\mathbf{l}}_{++} : \sum_l p_l = 1\}$. This set is the *unit simplex* in $\mathbf{R}^{\mathbf{l}}_{++}$. Because of Axiom 1, there is no loss of generality in assuming

$$Z : \Delta \to \mathbf{R}^{\mathbf{l}},$$

and we shall do so from here on out.

A.2. *For all* $p \in \mathbf{R}^{\mathbf{l}}{}_{++}$, $p \cdot Z(p) = 0$.

Axiom 2 is known as *Walras' law.* Why should it be true? For individuals, $p \cdot z_n(p) = 0$ is merely the claim that individual n consumes on her budget line, and Walras' Law follows from summing.

A.3. Z *is a continuous function on* $\operatorname{int} \Delta$*, or* Z *is a correspondence which is compact-, convex- and non-empty valued, and upper hemi-continuous.*

If demand is single-valued, then continuity, Axiom 3, is a natural assumption. The assumption is not required on the boundary since we are not assuming demand is well-defined when the price of a good is 0. When excess demand is a correspondence, what does *continuity* mean? For a function, continuity is the requirement that for every open set O, $Z^{-1}(O)$ is an open set. For a correspondence, there are two natural notions of inverse: Define $Z^w(O) = \{p : Z(p) \cap O\}$ and $Z_s(O) = \{p : Z(p) \subset O\}$. These are the *weak* and *strong* inverse of Z, respectively. The requirement that the weak inverse of an open set be open is called *lower hemi-continuity*, while the requirement that the strong inverse of an open set be open is called *upper hemi-continuity*. (An older literature uses the terminology semi-continuity, which unfortunately was already in play for something else.) A correspondence is *continuous* if it is both lower- and upper hemi-continuous. If a correspondence is singleton-valued, it is continuous if and only if, as a function, it is a continuous function.

A.4. *Z is bounded below; that is, there is a vector B such that $Z(p) \geq B$ for all $p \in \mathbf{R^n}_{++}$.*

If individuals can never demand negative amounts of a commodity, then excess demand is bounded below by $-e$, the aggregate endowment. This is Axiom 4. More generally, if we assume that consumer is budget sets are all subsets of a consumption set X_i and each X_i is bounded from below, then Axiom 4 will be satisfied.

A.5. *If p is a price vector such that for some good i, $p_i = 0$, then for every sequence of strictly positive prices p^n converging to p, $||Z(p)|| \to +\infty$.*

Finally, as the price of a commodity converges to 0, its demand becomes arbitrarily large. This is Axiom 5. Actually, it is stronger than Axiom 5, which only requires the demand for some commodity to diverge.

Theorem 1. *If excess demand $Z(p)$ satisfies Axioms 1–5 on $\mathbf{R^n}_{++}$, then there is a $p^* \neq 0$ such that $Z(p^*) = 0$.*

How does one prove such a theorem? The problem is a *fixed point problem*. The main tool is the following:

Theorem 2 (Brouwer). *If C is a compact, convex set and $f : C \to C$ is a continuous function, then there is an $x \in C$ such that $f(x) = x$.*

That is, x remains *fixed* under the action of f.

Theorem 3 (Kakutani). *If C is a compact, convex set and $F : C \to C$ is an upper hemi-continuous correspondence, then there is an $x \in C$ such that $x \in F(x)$.*

Proof. Suppose first that demand is a continuous function. Since demand is homogeneous of degree 0, prices can be normalized so that $\sum_i p_i = 1$. Let $\Delta = \{p \in \mathbf{R^n}_{++}\}$ such that $\sum_i p_i = 1$. This set is the strictly positive unit simplex in $\mathbf{R^n}$. For $\epsilon > 0$, define $\Delta_\epsilon = \{p \in \Delta : \forall i \; p_i \geq \epsilon\}$. These are compact, convex sets. Choose $n^{-1} > \epsilon > 0$. Since Z is bounded from below on Δ, $\max_i -Z_i(p) < B$ on Δ.

$$f_i^\epsilon(p) = \frac{p_i\big(1 + B^{-1}Z_i(p)\big) + \delta}{1 + n\delta}$$

where $\delta = \epsilon/(1 - n\epsilon)$. Let $f(p) = \big(f_i(p), \ldots, f_n(p)\big)$. The term $\big(1 + B^{-1}Z_i(p)\big)$ is positive. Thus on Δ, $f_i^\epsilon(p) > \epsilon$. Furthermore,

$$\begin{aligned}\sum_i f_i^\epsilon(p) &= \frac{1}{1+n\delta}\left(\sum_i p_1 + B^{-1}\sum_i p_i Z_i(p) + n\delta\right) \\ &= \frac{1}{1+n\delta}\Big(1 + n\delta\Big) \\ &= 1\end{aligned}$$

because of Walras' Law. Thus $f^\epsilon : \Delta \to \Delta_\epsilon$. It is continuous, and so Brouwer's Theorem implies that f^ϵ has a fixed point p^ϵ. Choose a sequence $\epsilon_k \to 0$. The sequence p^{ϵ_k} has a subsequence with a limit point p^*. We will see that p^* is a competitive equilibrium.

Clearly $p^* \geq 0$ and $\sum_i p_i^* = 1$. We also have that

$$p^{\epsilon_k} + n\delta_k p^{\epsilon_k} = p^{\epsilon_k} + B^{-1}Z_i(p^{\epsilon_k}) + \delta_k$$

First we need to show that $p_i^* > 0$. Suppose not. Then, taking limit superiors of both sides, $0 = \limsup B^{-1}Z_i(p^{\epsilon_k})$, but this violates Axiom 5. Finally, since $p^* \in \Delta$ and Z is continuous on Δ, $Z(p^*) = 0$.

For the correspondence case, define the correspondence F such that $q \in F^\epsilon(p)$ iff there is a $z \in Z(p)$ such that

$$q_i = \frac{p_i\big(1 + B^{-1}z_i\big) + \delta}{1 + n\delta}$$

This correspondence is clearly convex- and non-empty valued. For those who are interested, it is easy to show it is upper hemi-continuous. Thus according to Professor Kakutani (the father of Michiko Kakutani, for the literate among you), the correspondence F^ϵ has a fixed point p^ϵ. The proof now proceeds as before. $\square$

The relevant fact for proving that the excess demand correspondence is uhc, which we shall not prove here, concerns the graph of uhc correspondences.

Definition 3. *The* graph *of a correspondence* $F : X \to Y$ *is the set* $\{(x, y) : y \in F(x)\} \subset X \times Y$*.*

This is the same definition as that of the graph of a continuous function.

Lemma 1. *If* $F : X \to Y$ *is a correspondence and* Y *is compact, then* F *is uhc if and only if its graph is closed.*

The mathematically adept may enjoy proving this fact.

3 Private Ownership Economies

The private ownership economy is a general framework which fits together production and consumption. It locates consumption in individual consumers and production in individual firms. This is the most general framework in which we will examine the emergent properties of markets — the existence of price equilibria and their connection to optimal allocations.

The model supposes the existence of I consumers, J firms and L goods. Each consumer is characterized by a *preference order* $\succeq_i$ defined on a *consumption set* $X_i \subset \mathbf{R}^\mathbf{L}$, an *endowment bundle* $\omega_i \in \mathbf{R}^\mathbf{L}$, and a vector $\theta_i = (\theta_{ij})_{j=1}^J$ representing the *share of firm* j consumer i owns. Let $\omega = \sum_i \omega_i$ denote the *aggregate endowment* of the economy. Each firm is characterized by a *production set* $Y_j \subset \mathbf{R}^\mathbf{L}$. We adopt the convention that for any vector in a firm's production set, negative terms represent inputs and positive terms represent outputs. The ultimate reductionist act of economic theory is to define an economy thus:

Definition 4. *A* private ownership economy *is a tuple* $\left((X_i, \succeq_i, \theta_i, \omega_i)_{i=1}^I, (Y_j)_{j=1}^J\right)$.

This list contains all the data necessary to derive demand and supply functions, find equilibria, and so forth. The behavioral assumptions that go along with this are preference maximization by consumers and profit maximization by firms.

Note already the ways in which this model is restrictive.

1. Firms are assumed to profit maximize. What about alternative, "managerial" goals of firm behavior? How would models of "home production" fit into this framework.

2. There are no consumption externalities. This bites in several ways: First, i's preferences cannot be influenced by j's actions. There is no way to model your demand for air freshener on the consumption of your roommate's cigarettes. Second, individuals may have preferences not over individual consumption but over entire social states. Consumer is rankings of her own choices may be invariant to js consumption, but js consumption may nonetheless effect is utility. Third, i's consumption opportunities may be constrained by js choices.

3. There are no production externalities. You can imagine the list of possibilities.

A basic definition of the general competitive model is that of an allocation:

Definition 5. *An* allocation (x, y) *is a specification of a* consumption plan *for each consumer* i*, a vector* $x_i \in X_i$*, and a* production plan *for each firm* j*, a vector* $y_j \in Y_j$*. An allocation is* feasible *iff* $\sum_i x_i = \omega + \sum_j y_j$.

Following MWG, the set of feasible allocations is denoted by $A \subset \mathbf{R}^{\mathbf{L}(\mathbf{I}+\mathbf{J})}$. We will call x the *consumption allocation* and y the *production allocation* associated with the allocation $z = (x, y)$.

3.1 Competitive Equilibrium

One idea behind competitive equilibrium is that supply equals demand. But since we have a theory of where supply and demand come from, we are tempted to pierce the vale of Marshallian curves and understand competitive equilibrium directly in terms of the primitive, atomic economic concepts. Let $\mathcal{E} = \left((X_i, \succeq_i, \theta_i, \omega_i)_{i=1}^{I}, (Y_j)_{j=1}^{J}\right)$ denote a private ownership economy. I'm going to follow MWG here, but in fact one can do much better than this.

Definition 6. *A* competitive equilibrium *for the economy* $\mathcal{E}$ *is an allocation* (x^*, y^*) *and a price vector* p^* *such that*

1. *For every firm* j*,* y_j^* *maximizes profits among all feasible production plans in* Y_j*:*

$$p^* y_j^* \geq p^* y_j \quad \text{for all} \quad y_j \in Y_j.$$

2. *For every consumer i, x_i^* is preference-maximal among all affordable consumption plans. That is, $x_i^* \succeq_i x_i$ for all x_i in the set*

$$\{x_i : x_i \in X_i \quad \textit{and} \quad p^* x_i \le p^* \omega_i + \sum_j \theta_{ij} p^* y_j^*\}.$$

3. $(x^*, y^*) \in A$.

This definition is due to Debreu (see *Theory of Value*, ch. 5.5). Here is his existence theorem:

Theorem 4. *A competitive equilibrium for the private ownership economy $\mathcal{E}$ exists if for every i,*

1. X_i *is closed, convex and bounded from below,*
2. $\succeq_i$ *is* non-satiated *in* X_i,
3. *the relation* $\succeq_i$ *is* continuous,
4. *If* $x_i' \succ_i x_i$, *then for all* $0 < t < 1$, $tx_i' + (1-t)x_i \succ_i x_i$,
5. *there is an* x_i^0 *in* X_i *such that* $\omega_i \gg x_i^0$;

for every consumer j,

1. $0 \in Y_j$,
2. *the aggregate production set* $Y = \sum_j Y_j$ *is closed and convex,*
3. $Y \cap (-Y) = \emptyset$,
4. $Y \supset \mathbf{R^L}_-$.

Just to review, *non-satiation* means that for all $x_i \in X_i$ there is an $x_i' \in X_i$ such that $x_i' \succeq_i x_i$. *Continuity* means that for all x_i the sets $\{x_i' \in X_i : x_i' \succeq x_i\}$ and $\{x_i' \in X_i : x_i \succeq x_i'\}$ are closed in X_i. Conditions 2 and 4 together imply that preferences are locally non-satiated. Condition 5 for consumers is the 'cheaper point' assumption. Its purpose is to make sure that demand has the right continuity properties at the boundary of the consumption set. Suppose that a consumer's consumption set is $\mathbf{R^L}_+$ and her endowment is the the 0 vector. Imagine a sequence of prices in which the price of good 1, say, is always positive, but is converging to 0. Then demand for good 1 is always 0, but in the limit is could be strictly positive or even empty.

Continuity implies that the so-called 'better than' sets are open. The set $\{x_i' \in X_i : x_i' \succ_i x_i\}$ is the complement of the set $\{x_i' \in X_i : x_i \succeq_i x_i'\}$. Similarly for the 'worse than' sets.

For the firm, condition 3 states that production is not reversible. If the aggregate production plan y is feasible, then the plan $-y$ is not. Assumption 4 is free disposal.

Another useful concept is that of a *competitive equilibrium with transfers*. The idea is to find market clearing prices after we allow for arbitrary wealth transfers among consumers.

Definition 7. *A* competitive equilibrium with transfers *for the economy $\mathcal{E}$ is an allocation (x^*, y^*), a price vector p^* and an assignment of wealths $(w_1^*, \ldots, w_I^*)$ to consumers such that*

1. *For every firm j, y_j^* maximizes profits among all feasible production plans in Y_j:*

$$p^* y_j^* \geq p^* y_j \quad \text{for all} \quad y_j \in Y_j.$$

2. *For every consumer i, x_i^* is preference-maximal among all affordable consumption plans. That is, $x_i^* \succeq_i x_i$ for all x_i in the set*

$$\{x_i : x_i \in X_i \quad \text{and} \quad p^* x_i \leq w_i^*\}.$$

3. *$(x^*, y^*) \in A$.*

4. *$\sum_i w_i^* = \sum_i p^* \omega + \sum_j p^* y_j^*$.*

3.2 Pareto Optimality

The basic notion of social desirability is the *Pareto order*:

Definition 8. *A consumption plan x is* Pareto-better than *consumption plan x', written $x \succ_P x'$, iff for all i, $x_i \succeq_i x_i'$, and for some consumer k, $x_k \succ x_k'$. An allocation $z = (x, y)$ is* Pareto optimal *iff it is feasible, and if for no other feasible consumption plan $z' = (x', y')$ is it true that $x' \succ_P x$.*

How do we know an optimum exists? In exchange economies this is not hard. The set of feasible allocations is obviously compact, so suitable continuity assumptions on preferences should do the trick. When production is possible, compactness of the set of feasible allocations is not so obvious. Debreu (*Theory of Value*, Ch. 6.2.) gives us an answer.

Theorem 5. *The private ownership economy $\mathcal{E}$ has an optimum if*

1. *for all i, X_i is closed and bounded from below,*

2. *for every* $x'_i \in X_i$*, the set* $\{x_i \in X_i : x_i \succeq x'_i\}$ *is closed,*

3. $\sum_j Y_j$ *is closed, convex, and* $Y \int \mathbf{R^L}_+ = \{0\}$*, and*

4. $\omega \in \sum_i X_i - \sum_j Y_j$*.*

Proof. There are two parts to the proof. First, show that the set A of feasible allocations is compact, and then to show that a Pareto optimum exists. We will do half of the first part and all of the second.

To see that A is closed, let M denote $\{z = (z_1, \ldots, z_I, z_{I+1}, \ldots, z_{I+J}) \in \mathbf{R^{(I+J)L}} : \sum_{k=1}^{I+J} z_k = \sum_{i=1}^{I} \omega_i\}$. This is an affine subspace of $\mathbf{R^{(I+J)L}}$, hence closed. The set $M' = \prod_{i=1}^{I} X_i \times \prod_{j=1}^{J} Y_j \subset \mathbf{R^{(I+J)L}}$ is the product of closed sets, hence closed. Finally $A = M \cap M'$, and so is closed. Proving that A is bounded takes some apparatus that we are not going to introduce here.

To show that an optimum exists, define the relations $\succeq^i$ on M' such that $(x, y) \succeq^i (x', y')$ iff $x_i \succeq_i x'_i$. These relations inherit all the properties of $\succeq_i$. In particular, they are preference orders, and 'no worse than' sets are closed.

Let $O_1 = \cap_{z \in A}\{z' \in A : z' \succeq^1 z\}$. Each set in the intersection is compact because it is the intersection of a closed 'no worse than' set with the compact set A. Any finite intersection of theses sets is non-empty because $\succeq^1$ is transitive. If, for instance, $z_1 \succeq^1 z_2$, then $\{z' \in A : z' \succeq^1 z_1\} \cap \{z' \in A : z' \succeq^1 z_2\} = \{z' \in A : z' \succeq^1 z_1\}$. The finite intersection property of compact sets tells us that that O_1 is non-empty (and compact). Any allocation in O_1 is a feasible allocation which is best for consumer 1 among all feasible allocations.

Now repeat the same argument for consumer 2 on the set O_1. The outcome is a non-empty and compact set O_2 with the property that consumer 2 is getting an allocation which is preference-maximal for her among all allocations that are preference-maximal for consumer 1 on A. Repeating this successively for all consumers generates a set of allocations O_I for which each consumer i is getting preference-maximal allocations among those which are preference maximal for consumer $i - 1$ among those $\ldots$. These allocations are Pareto optimal. They are a peculiar set of Pareto optima, I admit, but demonstration of even one optimal allocation suffices to prove existence. □

The connection between equilibrium and optimality is subtle. The First Welfare Theorem gives conditions guaranteeing that a competitive equilibrium allocation is Pareto optimal. The Second Welfare Theorem guarantees that a Pareto optimal allocation is an equilibrium allocation for some endowment allocation. The conditions for the Second Theorem are stronger than the first, and the First is not automatic.

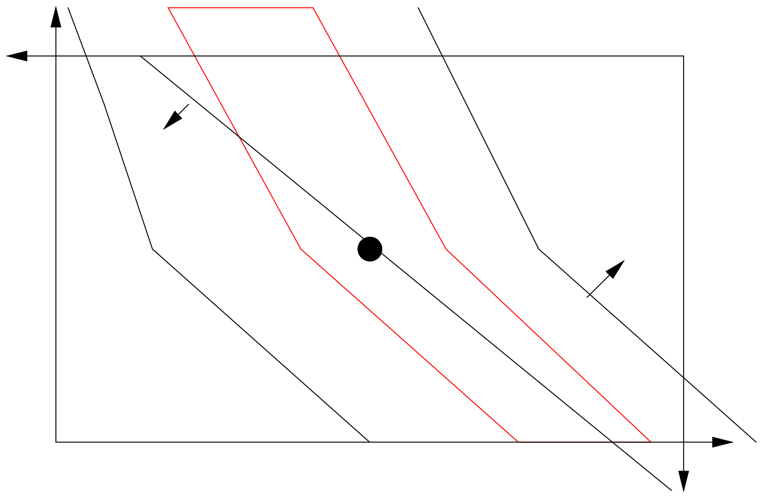

Figure 1: Failure of the First Welfare Theorem.

In Figure 1, the red polygon encloses a thick indifference curve of consumer A, whose origin is at the bottom left. The dot is an equilibrium allocation, and the straight line represents both an indifference curve for consumer B and the equilibrium budget sets of the two consumers. The equilibrium allocation is obviously not optimal.

Theorem 6 (First Welfare Theorem). *Let $\mathcal{E}$ be a private ownership economy with an equilibrium (p^*, x^*, y^*). Suppose for all i, $\succ_i$ is locally non-satiated at x_i^*. Then (x^*, y^*) is a Pareto-optimal allocation.*

Recall that a preference order $\succeq_i$ is *locally non-satiated* at x_i^* if in every open neighborhood of x_i^* there is an $x_i' \succ_i x_i^*$. Proving the First Welfare Theorem requires that in any equilibrium, any consumption bundle which is better for consumer i costs more. This is just what preference maximization on the budget set means. The proof requires more; specifically, than any bundle which is at least as good costs at least as much. This is exactly what fails in the example of Figure 1. The following lemma shows when this is true:

Lemma 1. *If $\succeq_i$ is locally non-satiated at bundle x_i^* which is preference-maximal on the set $\{x_i \in X_i : px_i \leq px_i^*\}$, and if $x_i' \succeq_i x_i^*$, then $px_i' \geq px_i^*$.*

Proof of Lemma 1. Since $\succeq_i$ is locally non-satiated at x_i^*, there is a sequence of consumption bundles x_i^n with limit x_i' such that $x_i^n \succ_i x_i'$. Transitivity implies that $x_i^n \succ_i x_i^*$. Preference maximality implies that $px_i^n > px_i^*$. Taking limits, $px_i' \geq px_i^*$. $\square$

Proof of the First Welfare Theorem. Suppose that (x', y') is Pareto-superior to (x^*, y^*). Then for all i, $x'_i \succeq_i x^*_i$, and for some individual this ranking is strict. This means that $p^* x'_i \geq p^* x^*_i$ for all i, with strict inequality for some i. Furthermore, for each j, $p^* y'_j \leq p^* y^*_j$ since each firm profit maximizes in equilibrium. Thus

$$p^* \omega = p^* \sum_i x^*_i - p^* \sum_j y^*_j < p^* \sum_i x'_i - p^* \sum_j y'_j.$$

The equality is a consequence of feasibility of the equilibrium allocation, and the inequality follows from the relations just established. Consequently, $\omega \neq \sum_i x'_i - \sum_j y'_j$. That is, the allocation (x', y') is not feasible. □

Notice that this argument requires no convexity whatsoever. The Second Welfare Theorem requires some convexity assumptions because, along the way, one has to prove the existence of an equilibrium. The welfare theorems are about the duality, in some sense, between Pareto optimality and competitive equilibrium. Unfortunately the duality is not perfect. The natural expression of duality for the Pareto problem is not competitive equilibrium, but a slightly different notion known in the literature as *quasi-equilibrium*.

Definition 9. *A* quasi-equilibrium *for the economy $\mathcal{E}$ is an allocation (x^*, y^*) and a price vector p^* such that*

1. *For every firm j, y^*_j maximizes profits among all feasible production plans in Y_j:*

$$p^* y^*_j \geq p^* y_j \quad \text{for all} \quad y_j \in Y_j.$$

2. *For every consumer i, x^*_i is expenditure-minimal on the 'no worse than' set. That is, $p^* x^*_i \leq x_i$ for all x_i in the set $\succeq_i (x^*_i)$.*

3. *$(x^*, y^*) \in A$.*

Quasi-equilibria are usually competitive equilibria because expenditure minimization is usually dual to preference maximization; but not always, and we will return to this after the proof of the Second Welfare Theorem.

It will be useful to have some additional notation for the statement and proof of the theorem. Let $\succ (x_i) = \{x'_i \in X_i : x'_i \succ_i x_i\}$ and $\succeq (x_i) = \{x'_i \in X_i : x'_i \succeq_i x_i\}$.

Theorem 7 (Second Welfare Theorem). *Let (x^*, y^*) be a Pareto Optimal allocation for a private ownership economy $\mathcal{E}$ with the properties that*

1. *for all i, X_i is convex,*
2. *the sets $\succeq (x_i^*)$ are convex,*
3. *for some consumer k, $\succ (x_k^*)$ is convex and $\succeq_k$ is locally non-satiated at x_k^*,*
4. *Y is convex.*

Then there is a p^* such that (x^*, y^*, p^*) is a quasi-equilibrium for $\mathcal{E}$.

The point of the Second Welfare Theorem is that competitive markets are unbiased. There are no Pareto optimal allocations which are unreachable by competitive markets or cannot be sustained by competitive markets after a suitable redistribution of income. That is to say, the only bias in the market is wealth.

Proof. Define the set $G = \sum_{i \neq k} \succeq_i (x_i^*) + \succ_k (x_k^*) - Y$. This set is convex and ω is not in G because the allocation is Pareto optimal. Thus there is a vector p^* such that $p^*\omega \leq p^*g$ for all $g \in G$. Since consumer k is locally non-satiated, there is a sequence of consumption plans x_k^n with limit x_k^*, each element of which is better for k than x_k^*. Then for all n the vector $g^n = \sum_{i \neq k} x_i^* + x_k^n - \sum_j y_j^*$ is in G, and the sequence g^n converges to $\sum_i x_i^* - \sum_j y_j^* = \omega$. Thus $p^*\omega = \inf\{p^*g : g \in G\}$, and we can conclude that $p^*x_i^*$ minimizes expenditure on the set $\succeq (x_i^*)$ for all $i \neq k$. This is also true for consumer k because of local non-satiation. (Why?) Also, $-p^*y_j^*$ is minimal on Y_j, which is to say that y_j^* is profit-maximal for firm j at price p^*. □

So far we have a feasible allocation and a price system such that firms profit maximize and consumers minimize expenditure on their 'no worse than' sets. This is the natural dual equilibrium solution to the Pareto optimization problem, and important enough that it has its own name: *quasi-equilibrium*. However it is not yet a competitive equilibrium. We need to move from expenditure minimization to utility maximization. This is the content of the next lemma.

Lemma 2. *Suppose that the preference order $\succeq_i$ has the property that for all $x_i \in X_i$, the set $\succ_i (x_i)$ is open. Suppose at a price p, x_i' minimizes expenditure on $\succeq_i (x_i')$. Suppose too that there is an $x_i^0 \in X_i$ such that $px_i^0 < px_i$. Then x_i' is preference-maximal on the set $\{x_i'' \in X_i : px_i'' \leq px_i'\}$.*

The existence of x_i' is referred to as the cheaper point assumption. Figure 2 demonstrates what can go wrong with the duality between expenditure minimization and utility maximization when the cheaper point assumption does not hold. In this figure, the consumption X is $\mathbf{R^2}_+$ in which the open triangle with vertices $(0,0)$, $(1,0)$ and $(0,1)$ has been removed. Prices and wealth are such that the

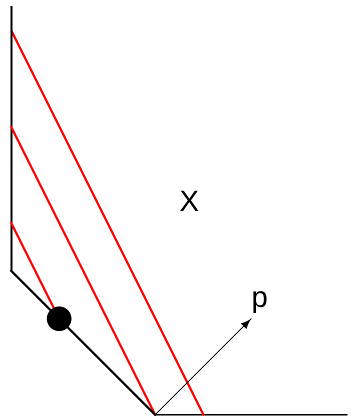

Figure 2: No cheaper point.

budget set is the lower 45 degree line. The indicated consumption bundle is expenditure minimizing on its 'no worse than' set, but it is not preference maximal on the budget set.

Proof of Lemma 2. Suppose contrary to the claim of the lemma that there was an $x_i''' \in \{x_i'' \in X_i : px_i'' \le px_i'\}$ such that $x_i''' \succ x_i'$. Since x_i' minimizes expenditure on the set $\succeq (x_i')$, it must be that $px_i''' = px_i'$. Since $\succ_i (x_i)$ is open, there is an open heighborhood N containing x_i''' such that for all $v \in N$, $v \succ_i x_i'$. Now consider the convex combinations $x(t) = tx_i^0 + (1-t)x_i'''$. For all $t > 0$, $px(t) < px_i'$. For t positive but sufficiently small, $x(t) \in N$, and so for these t, $x(t) \in \succ_i (x_i')$. This contradicts the hypotheses that x_i' is expenditure-minimizing. □

Finally, we can clarify the relationship between quasi- and competitive equilibrium.

Theorem 8. *Suppose that (x^*, y^*, p^*) is a quasi-equilibrium for the private ownership economy $\mathcal{E}$. Suppose that for all consumers i and for all $x_i \in X_i$, the set $\succ_i (x_i)$ is open. Then (x^*, y^*, p^*) is a competitive equilibrium with transfers.*

Proof. Take $w_i^* = p^* x_i^*$. The theorem is then a consequence of the definition of a quasi-equilibrium and the preceding Lemma. □

One might ask, is every quasi-equilibrium allocation Pareto-optimal? It clearly need not be, for exactly the reason illustrated in the picture. What does it take to get a a quasi-equilibrium allocation to be Pareto-optimal? The cheaper point assumption and open 'better than' sets; that is, precisely wwhen the quasi-equilibrium is a competitive equilibrium.

4 A Calculus Approach to the Welfare Theorems

It should be clear by now that the Welfare Theorems pose a duality of sorts. Take finding Pareto optima as a primal problem. The shadow prices for the feasibility constraints are the competitive equilibrium prices. This is especially clear in the proof of the Second Welfare Theorem, which identifies competitive equilibria as a supporting hyperplane, part of the dual description of G in the proofs of the last section. This interpretatiion becomes clearer to most people when we move to the more familiar ground of constrained optimization and Lagrange (Kuhn-Tucker) multipliers. The message is that equilibrium prices are shadow prices for the resource constraints in the Pareto optimization problem. This is most clearly seen in an exchange economy.

Suppose that each of I consumers has preferences which are represented by strictly concave, C^2 and strictly increasing utility functions $u_1, \ldots, u_I$ defined on X_i which is convex and has non-empty interior in $\mathbf{R}^\mathbf{L}$. That is, $D^2 u_i$ is negative definite and $Du_i \gg 0$ on X_i. Suppose too each consumer has a strictly positive endowment.

4.1 Optimality

If x^* is a Pareto optimal allocation, then there is no reallocation that can increase the utility of any consumer without decreasing the utility of anyone else. let $u_i(x_i^*) = u_i^*$. Then x^* solves the optimization problem on $\prod_i X_i$:

$$\begin{aligned} PO: \quad & \max\ u_1(x_1) \\ \text{s.t.} \quad & u_i(x_i) \geq u_i^* \quad \text{for } i = 2, \ldots, I, \\ & \sum_i x_i = \sum_i x_i^*. \end{aligned}$$

Since the u_i are strictly increasing, the weak inequalities can be assumed to be equalities. Let us, for simplicity, consider an allocation in which each x_i^* is interior to X_i^*.

The first order conditions are

$$\begin{aligned} Du_1(x_1^*) &= \lambda \\ \nu_i Du_i(x_i^*) &= \lambda. \end{aligned}$$

From these conditions the usual equality conditions for marginal rates of substitution follow. These conditions, along with the constraints, are sufficient for an allocation to be Pareto optimal.

4.2 Equilibrium

Now suppose an allocation $x'_1, \dots, x'_I$ is a competitive equilibrium at price vector p. Then $\sum_i x'_i = \sum_i \omega_i$, and for each i the bundle x'_i solves the optimization problem

$$\begin{aligned} CE_i : \qquad & \max\ u_i(x_i) \\ \text{s.t.} \quad & px_i \leq p\omega_i. \end{aligned}$$

Again one can take the inequality to be an equality. The first order conditions are

$$Du_i(x_i^*) = \eta_i p$$

These too are sufficient because of the concavity assumptions.

4.3 The Welfare Theorems

The proof of the welfare theorems amounts to showing:

First Welfare Theorem If for all i, x_i^*, η_i solves the first order conditions for CE_i with prices p, then x^* and $\lambda = \eta_1 p$, $\nu_i = \eta_1/\eta_i$ solves the PO first order conditions.

Second Welfare Theorem If x^*, ν and λ solve PO, then taking $\nu_1 = 1$, x^*, $\eta_i = 1/\nu_i$ and $p = \lambda$ solve all the CE_i first order conditions.

This is simple algebra.

Here is what one should take away from this exercise. The shadow price ratios on the resource constraints measure the relative scarcity of the commodities. And those price ratios are exactly the competitive price ratios. So in a competitive equilibrium, the equilibrium price indexes the scarcity of the commodities. It is an interesting exercise to carry this exercise out with production.